Contour Integration

Let $f(t)$ be a complex valued function of a real variable t on an interval $a \leq t \leq b$ (e.g. $f(t) = cost + isint$). We can then write:

$$f(t) = u(t) + iv(t)$$

where $u(t)$ and $v(t)$ are real valued functions.

Def. $f(t)$ is said to be **integrable** on $a \leq t \leq b$ if both $u(t)$ and $v(t)$ are integrable on $a \leq t \leq b$. In that case we define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

The usual rules of integration of real valued functions apply. In particular, the two forms of the Fundamental Theorem of Calculus hold:

If $f(t)$ is a continuous function then:

$$\frac{d}{dt}\int_a^t f(x)dx = f(t)$$

and if $f'(t)$ is continuous then:

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Now we want to extend the notion of integration to the integration of a function $f(t)$ on a curve in the complex plane.

We can describe a curve in $\mathbb{C}$ by a parametrization:

$$z(t) = x(t) + iy(t); \qquad a \leq t \leq b$$

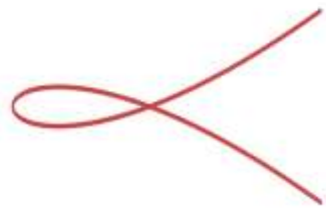
(This is similar to parametrizing a curve in $\mathbb{R}^2$ by $\vec{c}(t) =< x(t), y(t) >$).

We say the curve $z(t)$ is continuous/differentiable if $x(t)$ and $y(t)$ are continuous/differentiable.

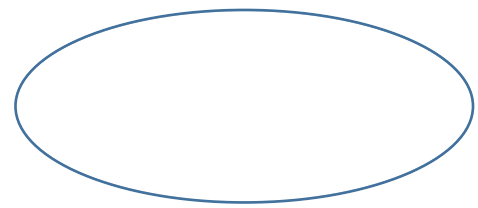
Def. We say the curve C represented by $z(t)$ is a **simple curve** if $z(t_1) \neq z(t_2)$ for any distinct $t_1, t_2 \in [a, b]$, except we will allow $z(a) = z(b)$.

This ensures that a simple curve C does not intersect itself. If $z(a) = z(b)$ we say that C is a **simple closed curve (or just a closed curve) or a Jordan curve**.

Simple Curve

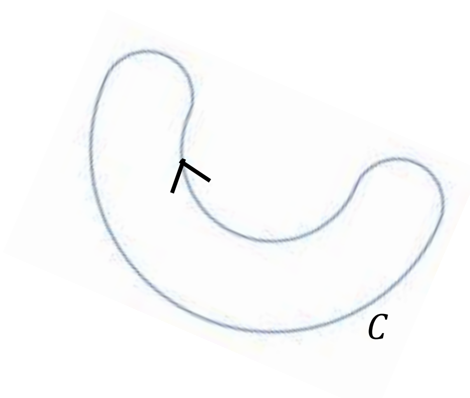


Not a Simple Curve

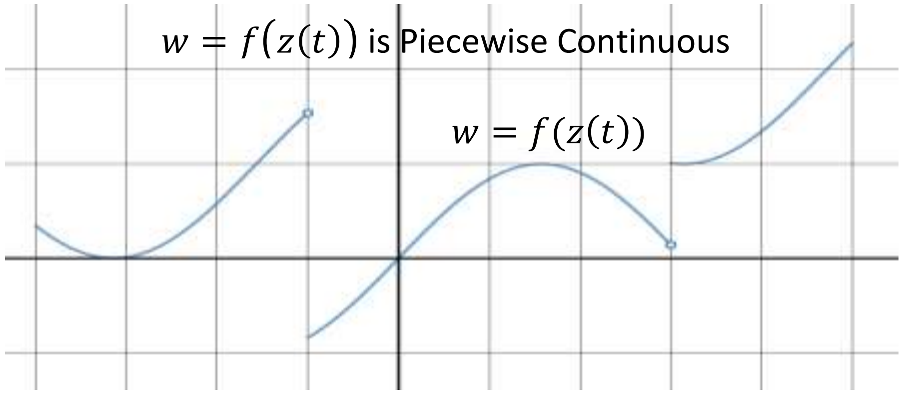


Simple Closed Curve

If C is a closed curve, we take the positive direction to be counterclockwise (i.e. if you are walking around the curve, the region bounded by the curve remains to your left). We will assume all closed curves are oriented in the positive direction.

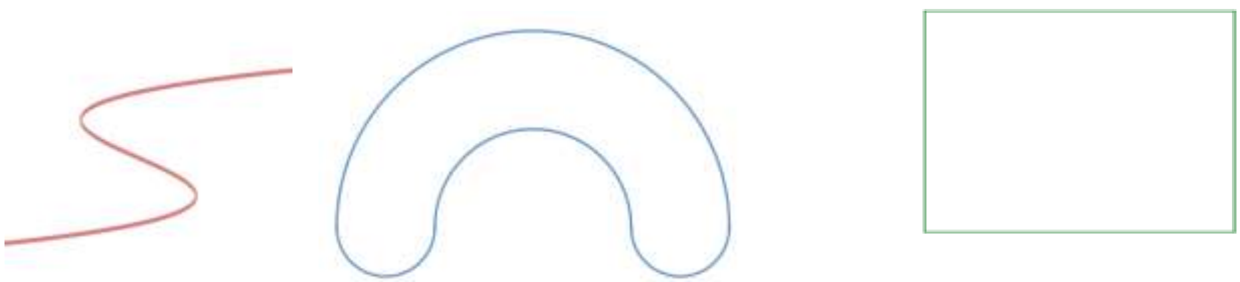


Def. The function $f(z)$ is said to be **continuous on C** if $f(z(t))$ is a continous function for $a \leq t \leq b$. $f(z)$ is said to be **piecewise continuous on $a \leq t \leq b$** if $[a, b]$ can be broken up into a finite number of subintervals such that $f(z)$ is continuous on each subinterval.



Def. A **smooth arc** (or curve) C is one in which $z'(t)$ is continuous on $a \leq t \leq b$.

Def. A **contour** is an arc consisting of a finite number of connected smooth curves, i.e. a contour is a piecewise smooth curve.



Examples of Contours

A simple closed contour is called a **Jordan contour**.

Def. We define the **contour integral** of a piecewise continuous function on a contour C by:

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)z'(t)dt.$$

This is very similar to a line integral of a vector field, however, the multiplication of $f(z(t))$ and $z'(t)$ is done by the usual multiplication of complex numbers, where the multiplication that occurs in a line integral of a vector field is the dot product of two vectors (which is different).

As is true of line integrals of vector fields, $\int_C f(z)dz$ does not depend on the parametrization of C as long as the orientation is preserved.

The usual properties of line integrals hold:

1. $\int_C (\alpha f(z) + \beta g(z))\, dz = \alpha \int_C f(z)dz + \beta \int_C g(z)dz;$
   $\alpha, \beta \in \mathbb{C}, f(z), g(z)$ are piecewise continuous.

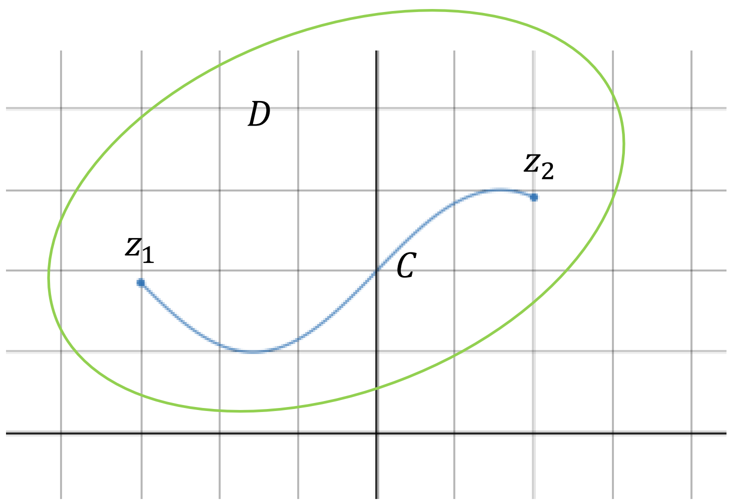
2. If we reverse the orientation of C then: $\int_{-C} f(z)dz = -\int_C f(z)dz$

3. If C=$C_1 + C_2$ is the sum of contours C_1 and C_2 then
   $$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

Theorem: Suppose $F(z)$ is analytic and $f(z) = F'(z)$ is continuous in a domain D. For a contour C lieing inside D with endpoints $z_1,\ z_2$

$$\int_C f(z)dz = F(z_2) - F(z_1).$$

Proof:



$$\int_C f(z)dz = \int_C F'(z)dz = \int_a^b F'\big(z(t)\big)z'(t)dt;$$

where $z(t)$ is a parametrization of the curve C with $z(a) = z_1$ and $z(b) = z_2$.

$$\begin{aligned}\int_C f(z)dz &= \int_a^b F'\big(z(t)\big)z'(t)dt = \int_a^b \frac{d}{dt}\big[F\big(z(t)\big)\big]dt \\ &= F\big(z(b)\big) - F(z(a)) \\ &= F(z_2) - F(z_1).\end{aligned}$$

1. As a consequence of this theorem, for a closed contour C

$$\oint_C f(z)dz = \oint_C F'(z)dz = 0$$

where $\oint_C$ denotes an integral over a closed contour.

2. Notice this theorem says that if $f(z) = F'(z)$ then $\int_C f(z)dz$ depends only on the endpoints of C. So any contour with the same endpoints will result in the same value for the contour integral.

Sometimes one evaluates a complex integral by reducing it to two real line integrals.

$f(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$, then we have:

$$\int_C f(z)dz = \int_C (u(x,y) + iv(x,y))(dx + idy)$$

$$= \int_C [(udx - vdy) + i(vdx + udy)].$$

Ex. Evaluate $\int_C 3z^2dz$ where C is a line segment from 0 to $1+i$.

Here we can use the fact that $\frac{d}{dz}(z^3) = 3z^2$ so

$$\int_C 3z^2dz = z^3\Big|_{z=0}^{z=1+i} = (i+1)^3 - 0^3 = 1 + 3i + 3i^2 + i^3$$

$$= 1 - 3 + 3i - i = -2 + 2i.$$

Or If we wanted to do this by parametrizing the line segment we could say:

The line segment C is given by: $z(t) = t + ti \qquad 0 \le t \le 1$

$$z'(t) = 1 + i$$

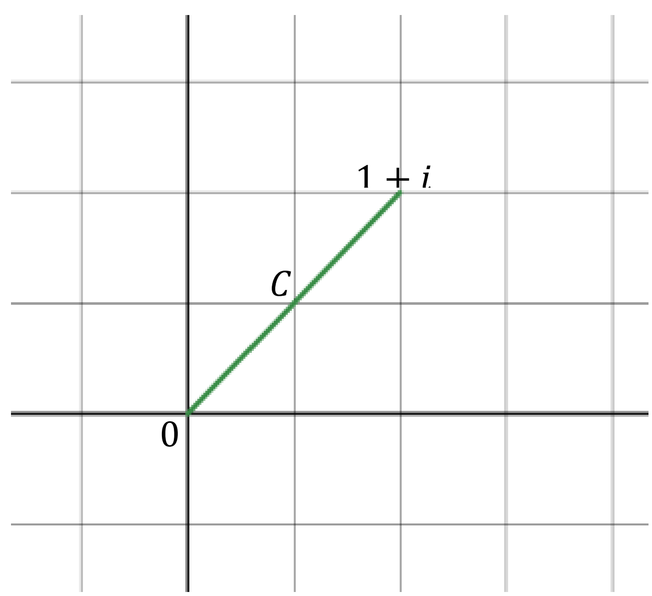
$$\int_C 3z^2dz = \int_{t=0}^{t=1} 3(t+ti)^2(1+i)dt$$

$$= \int_{t=0}^{t=1} 3(t^2 + 2t^2i - t^2)(1+i)dt$$

$$= \int_{t=0}^{t=1} 6t^2i(1+i)dt$$

$$= 2t^3i(1+i)\Big|_{t=0}^{t=1}$$

$$= 2i(1+i) = -2 + 2i.$$



Note: You can always parametrize a line segment from $z_1 = a + bi$ to $z_2 = c + di$ by:

$$z(t) = z_1 + t(z_2 - z_1) \qquad 0 \le t \le 1$$

$$z(t) = (a + bi) + t[(c - a) + (d - b)i]$$

$$= (a + (c - a)t) + (b + (d - b)t)i \qquad 0 \le t \le 1.$$

$$z'(t) = (c - a) + (d - b)i.$$

Given any smooth curve in $\mathbb{C}$ there's an infinite number of ways to parametrize it. Below is a set of parametrizations of some common curves.

Parametrizations of Some Common Curves in $\mathbb{C}$

Curve	Parametrization
Line segment from $z_1 = a + bi$ to $z_2 = c + di$	$z(t) = z_1 + t(z_2 - z_1)$; $0 \le t \le 1$
Circle of radius R and center $z = 0$	$z(t) = Re^{it}$; $0 \le t \le 2\pi$
Circle of radius R and center $z = a$	$z(t) = a + Re^{it}$; $0 \le t \le 2\pi$
Curve $y = f(x)$, $a \le x \le b$	$z(t) = t + if(t)$; $a \le t \le b$
Curve given by $\gamma(t) = \big(x(t), y(t)\big)$, $a \le t \le b$	$z(t) = x(t) + iy(t)$; $a \le t \le b$

Ex. Evaluate $\oint_C z^n dz$ where n is an integer and C is the unit circle $|z| = 1$.

$$z(t) = e^{it} \quad 0 \le t \le 2\pi, \qquad\qquad z'(t) = ie^{it}dt$$

$$\oint_C z^n dz = \int_0^{2\pi} \left(e^{it}\right)^n \left(ie^{it}\right)dt = \int_0^{2\pi} e^{i(n+1)t} i dt$$

$$= \left. \frac{ie^{i(n+1)t}}{i(n+1)} \right|_{t=0}^{t=2\pi} = 0; \qquad \text{if } n \ne -1$$

$$= it|_{t=0}^{t=2\pi} = 2\pi i \qquad \text{if } n = -1.$$

Ex. Evaluate $\int_C |z|^2 dz$ where C is:

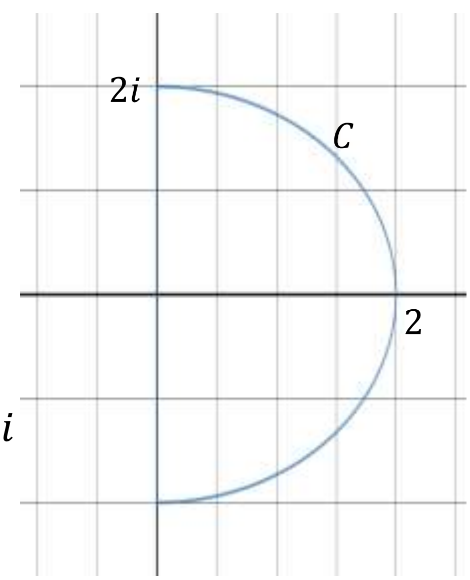
a. The right half of the circle $|z| = 2$ (i.e. $x^2 + y^2 = 4,\ x \ge 0$)
b. The circle $|z - 1| = 1$ (circle of radius 1, center $z = 1$)
c. The right triangle with vertices at $0, 2$, and $2 + 2i$.

a. $z(t) = 2e^{it} \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}, \qquad\qquad z'(t) = 2ie^{it}dt$

$$\int_C |z|^2 dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|2e^{it}\right|^2 \left(2ie^{it}\right)dt$$

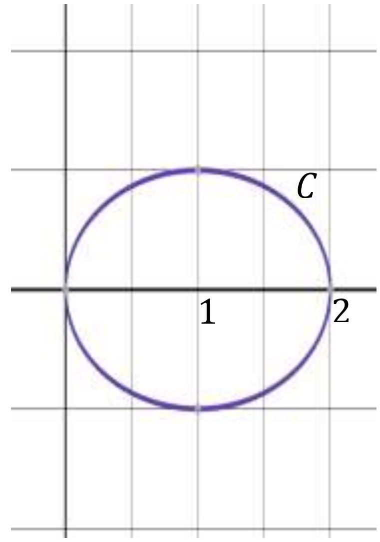
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\left(2ie^{it}\right)dt = 8e^{it}\Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}$$

$$= 8\left(e^{\frac{i\pi}{2}} - e^{-\frac{i\pi}{2}}\right) = 8\big(i - (-i)\big) = 16i$$



b. $z(t) = 1 + e^{it}; \quad 0 \le t \le 2\pi,$ $\qquad z'(t) = ie^{it}dt$

$$\oint_C |z|^2 dz = \int_0^{2\pi} \left|1 + e^{it}\right|^2 \left(ie^{it}\right)dt.$$



$$\begin{aligned}\left|1 + e^{it}\right|^2 &= \left(1 + e^{it}\right)\left(\overline{1 + e^{it}}\right) \\ &= (1 + e^{it})(1 + e^{-it}) \\ &= 1 + e^{it} + e^{-it} + 1 \\ &= 2 + 2cost.\end{aligned}$$

so:

$$\oint_C |z|^2 dz = \int_0^{2\pi} (2 + 2cost)(icost - sint)dt$$

$$= \int_0^{2\pi} (-2sint - 2(cost)(sint) + i(2cost + 2cos^2 t)dt$$

$$\oint_C |z|^2 dz = (2cost + cos^2 t + 2isint)|_{t=0}^{t=2\pi} + 2i\int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}cos2t\right)dt$$

$$= 0 + 2i(\frac{1}{2}t + \frac{1}{4}sin2t)\Big|_{t=0}^{t=2\pi} = 2\pi i.$$

c. Parametrize the triangle as 3 line segments.

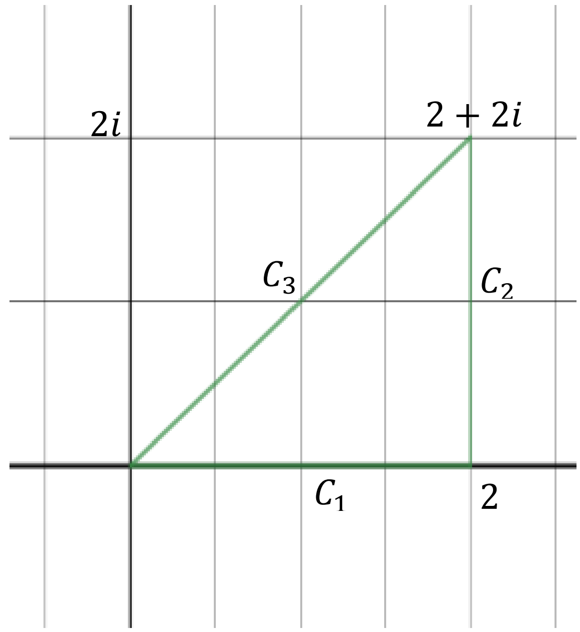
$z_1(t) = 0 + t(2 - 0) = 2t \qquad 0 \le t \le 1 \qquad z_1{}'(t) = 2$

$z_2(t) = 2 + t(2 + 2i - 2) = 2 + 2ti \quad 0 \le t \le 1 \qquad z_2{}'(t) = 2i$

$z_3(t) = 2 + 2i + t(0 - (2 + 2i))$

$\qquad = (2 - 2t) + i(2 - 2t) \qquad 0 \le t \le 1 \quad z_3{}'(t) = -2 - 2i.$

$$\oint_C |z|^2 dz = \int_{C_1} |z|^2\, dz + \int_{C_2} |z|^2\, dz + \int_{C_3} |z|^2\, dz$$



$$\int_{C_1} |z|^2\, dz = \int_0^1 |2t|^2 (2) dt = \int_0^1 8t^2 dt$$

$$= \frac{8}{3} t^3 \Big|_{t=0}^{t=1} = \frac{8}{3}\,.$$

$$\int_{C_2} |z|^2\, dz = \int_0^1 |2 + 2ti|^2 (2i) dt$$

$$= \int_0^1 (4 + 4t^2)(2i) dt$$

$$= 2i\left(4t + \frac{4}{3} t^3\right)\Big|_{t=0}^{t=1}$$

$$= 2i\left(4 + \frac{4}{3}\right) = \frac{32}{3} i\,.$$

$$\int_{C_3} |z|^2\, dz = \int_0^1 |(2 - 2t) + (2 - 2t)i|^2 (-2 - 2i) dt$$

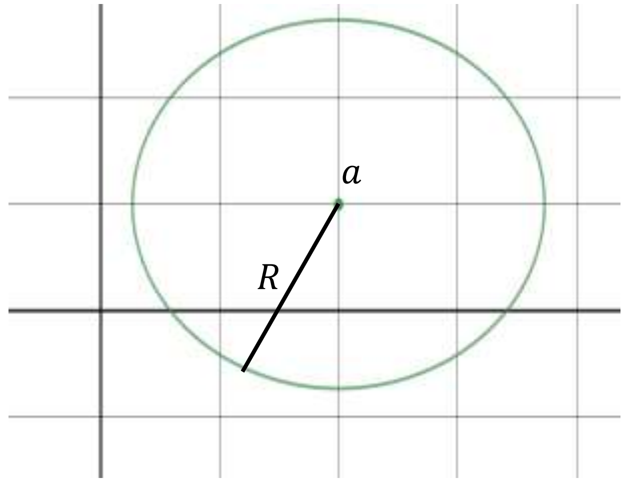
$$= (-2 - 2i) \int_0^1 2(2 - 2t)^2 dt$$

$$= (-2 - 2i)(-\frac{1}{3}(2 - 2t)^3 \Big|_{t=0}^{t=1}$$

$$= (-2 - 2i)\left(\frac{8}{3}\right) = -\frac{16}{3} - \frac{16}{3} i.$$

$$\oint_C |z|^2 dz = \frac{8}{3} + \frac{32}{3} i - \frac{16}{3} - \frac{16}{3} i = -\frac{8}{3} + \frac{16}{3} i.$$

Ex. Evaluate $\oint_C (z-a)^n dz$, where n is an integer and C is the circle of radius R around $a \in \mathbb{C}$ (i.e. $|z-a| = R$).



The easiest way to do this is to substitute
$w = z - a,\ dw = dz.$
Then the integral becomes:

$\oint_C (w)^n dw$, where C is $|w| = R$.
This is an integral we did earlier.

$$\oint_C (z-a)^n dz = \oint_C (w)^n dw = 0 \qquad \text{if } n \neq -1$$

$$= 2\pi i \qquad \text{if } n = -1.$$

Or we could note that: $(z-a)^n = \frac{d}{dz}\left(\frac{1}{n+1}(z-a)^{n+1}\right)$ if $n \neq -1$ so

$$\oint_C (z-a)^n dz = F(z_2) - F(z_1) = 0; \quad \text{since } C \text{ is a closed curve.}$$

For $n = -1$ then $(z-a)^{-1} = \frac{d}{dz}(\log(z-a))$, but $\log(z)$ is not single valued on any domain that contains the circle $|z-a| = R$. However, we can evaluate this as a line integral:

$$z(t) = a + Re^{it},\ 0 \le t \le 2\pi; \qquad\qquad z'(t) = iRe^{it}$$

$$\oint_C \frac{1}{z-a} dz = \int_0^{2\pi} \frac{1}{Re^{it}} \left(iRe^{it}\right) dt$$

$$= \int_0^{2\pi} i\,dt$$

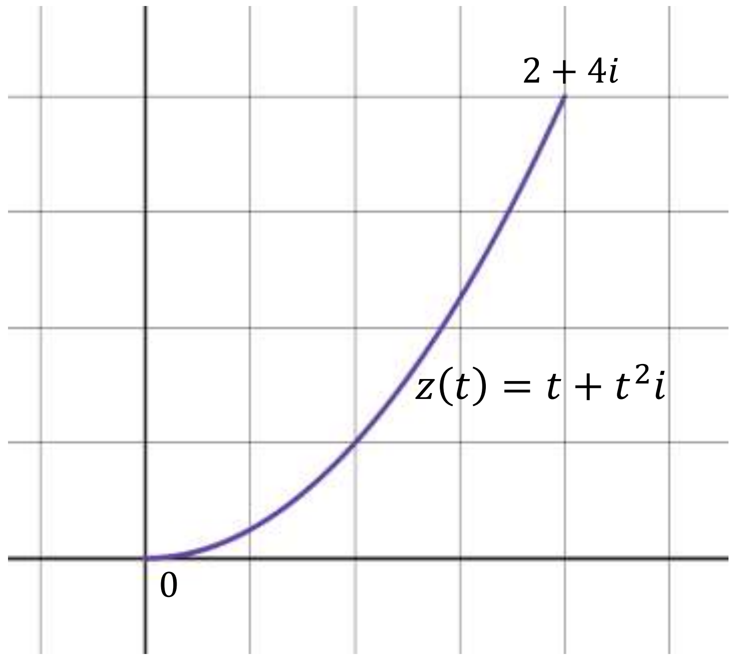
$$= 2\pi i.$$

Ex. Evaluate $\int_0^{2+4i} Re(z)dz$; on the parabola $y = x^2$.

The curve $y = x^2$ is represented by the points $(t, t^2),\ 0 \le t \le 2$ in $\mathbb{R}^2$ and by

$$z(t) = t + t^2 i \quad 0 \le t \le 2 \text{ in } \mathbb{C}.$$

So $z'(t) = 1 + 2ti$.



$Re(z) = x = t$; so the integral becomes;

$$\int_0^{2+4i} Re(z)dz = \int_{t=0}^{t=2} t(1 + 2ti)dt$$

$$= \int_{t=0}^{t=2} (t + 2it^2)dt$$

$$= \left(\frac{t^2}{2} + \frac{2i}{3}t^3\right)\Big|_{t=0}^{t=2}$$

$$= 2 + \frac{16}{3}i\ .$$

Theorem: Let $f(z)$ be continuous on a contour C. Then:

$$\left|\int_C f(z)dz\right| \leq ML$$

Where L is the length of C and $|f(z)| \leq M$ for $z \in C$.

Proof: $\int_C f(z)dz = \lim_{\Delta z \to 0} \sum_{i=1}^{n} f(z_i)\Delta z$ (equivalent to earlier definition)

$$\left|\int_C f(z)dz\right| = \left|\lim_{\Delta z \to 0} \sum_{i=1}^{n} f(z_i)\Delta z\right|$$

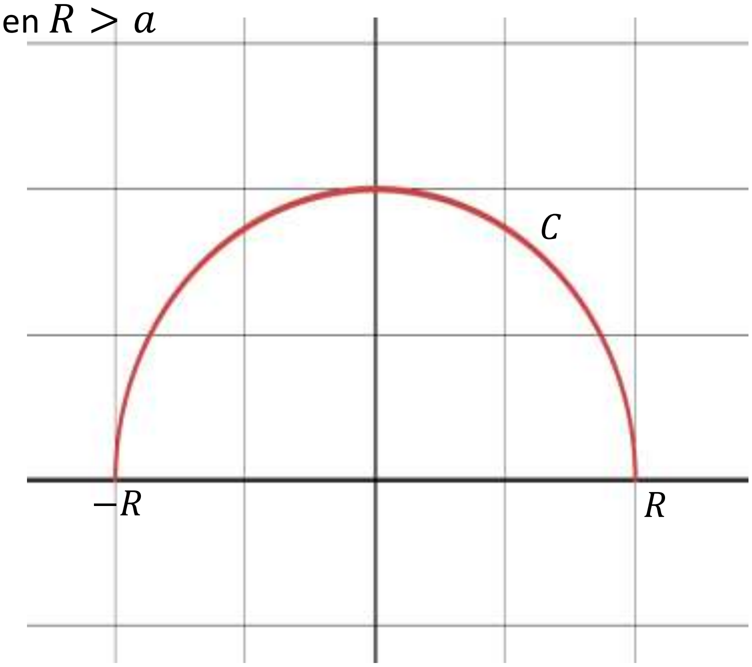
$$\leq \lim_{\Delta z \to 0} \sum_{i=1}^{n} |f(z_i)\Delta z| \qquad \text{(triangle inequality)}$$

$$\leq \lim_{\Delta z \to 0} \sum_{i=1}^{n} M|\Delta z| \qquad (|f(z)| \leq M \text{ for } z \in C)$$

$$= ML \qquad \left(\lim_{\Delta z \to 0} \sum_{i=1}^{n} |\Delta z| = L\right).$$

Ex. Let C be an open upper semicircle of radius R with its center at the origin. Let $f(z) = \frac{1}{z^2+a^2}$; $a \in \mathbb{R}$, $a > 0$. Show that

1. $|f(z)| \leq \frac{1}{R^2-a^2}$; when $R > a$
2. $\left|\int_C f(z)dz\right| \leq \frac{\pi R}{R^2-a^2}$; when $R > a$
3. $\lim_{R \to \infty} \int_C f(z)dz = 0.$



1. From the triangle inequality we know that: $\big||z_1| - |z_2|\big| \le |z_1 + z_2|$.

Apply this inequality to $z_1 = z^2$ and $z_2 = a^2$: $\big||z^2| - |a^2|\big| \le |z^2 + a^2|$.

If z is on the upper semicircle of radius R, $z = Re^{it}$, $0 < t < \pi$, so

$\Big||R^2e^{2it}| - |a^2|\Big| \le |z^2 + a^2|$ or equivalently: $|R^2 - a^2| \le |z^2 + a^2|$.

Since $0 < a < R$, $0 \le R^2 - a^2 \le |z^2 + a^2|$ or

$$\frac{1}{R^2-a^2} \ge \frac{1}{|z^2+a^2|} = |f(z)|.$$

2. Since $f(z)$ is continuous on C, we have: $\left|\int_C f(z)dz\right| \le ML$.

   We can use $M = \frac{1}{R^2-a^2}$ from part #1 (when $a < R$) , and we know that $L = \pi R$, since C is a semicircle. Thus we have:

   $$\left|\int_C f(z)dz\right| \le ML = \frac{\pi R}{R^2-a^2}; \quad \text{when } a < R.$$

3. $$0 \le \left|\lim_{R\to\infty} \int_C f(z)dz\right| \le \lim_{R\to\infty} \frac{\pi R}{R^2-a^2} = 0; \text{ so}$$

$$\lim_{R\to\infty} \int_C f(z)dz = 0 \text{ by the squeeze theorem.}$$