Multivalued Functions

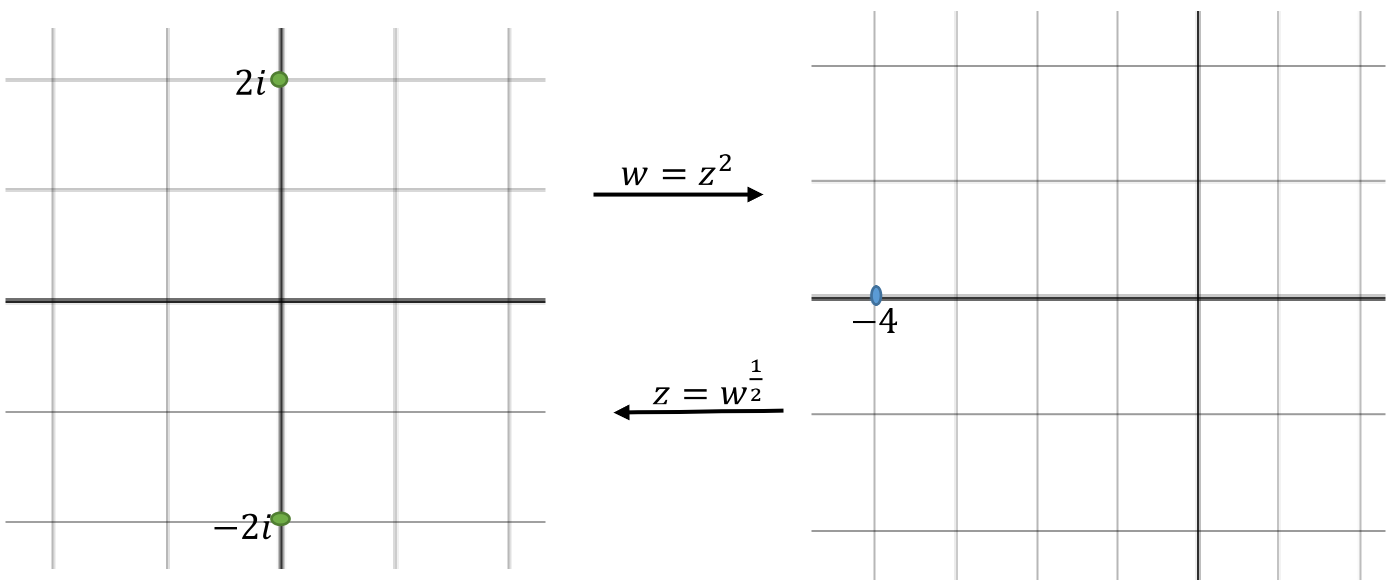
A single valued function $w = f(z)$ gives one value w for a given value of z.

Ex. $w = z^2$. For example, if $z = 2i$ then $w = (2i)^2 = -4$.

A multivalued function yields more than one value w for a given z.
Multivalued functions frequently arise as the inverse of a single valued function.

Ex. If $w = z^2$, it's inverse $z = w^{\frac{1}{2}}$ is a multivalued function. If $w = -4$ then

$$z = (-4)^{\frac{1}{2}} = \pm 2i.$$



We can represent any complex number $w = x + iy$ in polar form by $w = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$, and $\theta = \theta_p + 2\pi n$, n an integer and $0 \le \theta_p < 2\pi$. So

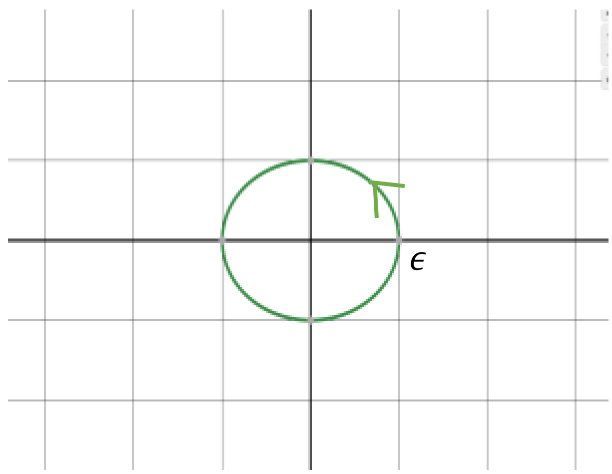
$$z = w^{\frac{1}{2}} = (re^{i\theta})^{\frac{1}{2}} = (re^{i(\theta_p + 2\pi n)})^{\frac{1}{2}} = r^{\frac{1}{2}}\left(e^{\frac{i\theta_p}{2}}\right)\left(e^{n\pi i}\right); \quad n \text{ an integer.}$$

For any non-zero value of w, $w^{\frac{1}{2}}$ has two values:

$$w^{\frac{1}{2}} = (\sqrt{r})e^{\frac{i\theta_p}{2}} \quad \text{and} \quad w^{\frac{1}{2}} = (\sqrt{r})e^{\frac{i\theta_p}{2}}e^{\pi i} = -(\sqrt{r})e^{\frac{i\theta_p}{2}}.$$

If we let n be any integer other than $n = 0{,}1$, we would simply repeat values. For example, if $n = 2$, $\quad w^{\frac{1}{2}} = r^{\frac{1}{2}}\left(e^{\frac{i\theta_p}{2}}\right)\left(e^{2\pi i}\right) = (\sqrt{r})e^{\frac{i\theta_p}{2}}$; since $e^{2\pi i} = 1$.

Notice that something very odd happens when we let w traverse a small circle of radius $\epsilon > 0$ around the point $w = 0$. We will see that we won't return to the same value of $z = w^{\frac{1}{2}}$ as we let w traverse the circle.



Let's start at the point $w = \epsilon$, if $w = re^{i\theta}$, then $r = \epsilon$ and $\theta = 0$. Taking the part of $z = w^{\frac{1}{2}} = (\sqrt{\epsilon})e^{\frac{i\theta}{2}}$, let θ go from 0 to 2π. w is the same point for $\theta = 0$ or $\theta = 2\pi$ (we've just gone around the cirlce), but

$$\theta = 0, \qquad z = w^{\frac{1}{2}} = (\sqrt{\epsilon})e^{\frac{i\theta}{2}} = \sqrt{\epsilon}$$

$$\theta = 2\pi, \qquad z = w^{\frac{1}{2}} = (\sqrt{\epsilon})e^{\frac{i\theta}{2}} = (\sqrt{\epsilon})e^{\pi i} = -\sqrt{\epsilon}$$

So we get 2 different values to this function for the same point w.

Def. A **branch point** of a multivalued function is one where upon traversing a small circle around the point, the value of the function does not return to its original value.

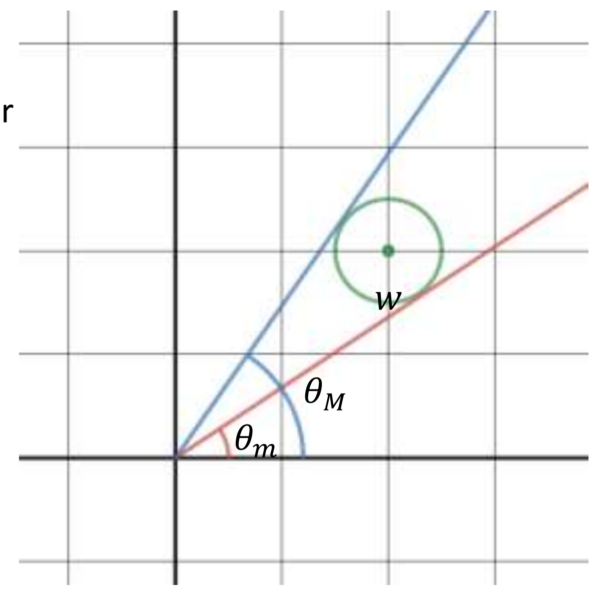
The point $w = 0$ is a branch point for the function $z = w^{\frac{1}{2}}$.

In this example, $w = \infty$ is also a branch point. We can see this by substituting $w = \frac{1}{t}$ into the function and studying the point where $t = 0$.

$$z = w^{\frac{1}{2}} = \frac{1}{t^{\frac{1}{2}}} = t^{-\left(\frac{1}{2}\right)}.$$

A similar argument to the one we just made for $z = w^{\frac{1}{2}}$ around $w = 0$, also shows that $t = 0$ is a branch point for $z = t^{-\frac{1}{2}}$.

Notice that if we take any other point $w \neq 0 \; or \; \infty$, it will not be a branch point for $z = w^{\frac{1}{2}}$.



For example, in the example to the right for any point on the green circle about w represented by $re^{i\theta_p}$, we have a minimum theta, θ_m, and a maximum theta, θ_M, such that: $0 < \theta_m \leq \theta_p \leq \theta_M < \frac{\pi}{2} < 2\pi$.

So when we go around the circle θ_p can't go from θ to $\theta + 2\pi$. In fact, θ_p will return to its original value as we go around the circle.

Thus $z = w^{\frac{1}{2}}$ will return to its original value as we go around the circle. Thus $w \neq 0 \; or \; \infty$ will not be a branch point.

When working with multivalued functions one generally tries to find a subset of $\mathbb{C}$ where the function is single valued and continuous. A continuous single valued function obtained by restricting a multivalued function to a subset of $\mathbb{C}$ is called a **branch** of the multivalued function.

For $z = w^{\frac{1}{2}}$ we can do this by removing the positive real axis (along with $z = 0$ and $z = \infty$) from $\mathbb{C}$. The positive real axis in this case is called a **branch cut**. In this particular case any ray starting at 0 given by $\theta = constant$ would also work as a branch cut.

When we take $n = 0$, and $0 \leq \theta_p < 2\pi$ in the formula:

$$z = w^{\frac{1}{2}} = (re^{i(\theta_p + 2\pi n)})^{\frac{1}{2}}$$

it's called the **principal value** of $z = w^{\frac{1}{2}}$.

Ex. Find all possible values for $(1 + \sqrt{3}i)^{\frac{1}{2}}$ and identify its principal value.

First convert $1 + \sqrt{3}i$ to polar form.

$w = 1 + \sqrt{3}i, \quad x = 1, \ y = \sqrt{3}$,

so $r = \sqrt{1+3} = 2, \ \ tan\theta = \left(\frac{\sqrt{3}}{1}\right); \ so \ \ \theta = \frac{\pi}{3}$, since w is in the first quadrant.

$w = 2e^{(\frac{\pi i}{3} + 2n\pi i)}$ so $w^{\frac{1}{2}} = (2e^{(\frac{\pi i}{3} + 2n\pi i)})^{\frac{1}{2}}$.

$$n = 0 \ \ (principal\ value) \ \ w^{\frac{1}{2}} = (2e^{\left(\frac{\pi i}{3}\right)})^{\frac{1}{2}} = \sqrt{2}e^{\frac{\pi i}{6}} = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right).$$

$n = 1$

$$w^{\frac{1}{2}} = (2e^{\left(\frac{\pi i}{3}+2\pi i\right)})^{\frac{1}{2}}$$
$$= \sqrt{2}e^{\left(\frac{\pi i}{6}+\pi i\right)}$$
$$= -\sqrt{2}e^{\left(\frac{\pi i}{6}\right)}$$
$$= -\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right).$$

Ex. Find branch points and branch cuts for

a. $w = (z - z_0)^{\frac{1}{2}}; \qquad z_0 \in \mathbb{C}$
b. $w = (az + b)^{\frac{1}{2}}; \quad a, b \in \mathbb{C}, a \neq 0.$

a. Let $z' = z - z_0$, then the function becomes $w = (z')^{\frac{1}{2}}$. We just saw that this function has branch points at $z' = 0$ and $z' = \infty$. Thus

$w = (z - z_0)^{\frac{1}{2}}$ has branch points at $0 = z' = z - z_0$ or $z = z_0$ and $z' = \infty$ or $z = \infty$. We could use as a branch cut the ray starting at $z' = 0$ and going out the positive x' axis. This is the same as taking a ray starting at $0 = z' = z - z_0$ or a ray starting at $z = z_0$ and going parallel to the positive real axis.

b. Let $z' = az + b$ then the function again becomes $w = (z')^{\frac{1}{2}}$. So $z' = 0$ and $z' = \infty$ are branch points. So $0 = z' = az + b$ means that $z = -\frac{b}{a}$ is a branch point and $z = \infty$ is a branch point. As a branch cut we can take a ray starting at $z = -\frac{b}{a}$ going to ∞ running parallel to the real axis (actually any ray starting from $z = -\frac{b}{a}$ going to ∞ will work).

Let's consider the inverse function of $z = e^w$. Letting $w = u + iv$ we have:

$$z = e^w = e^{(u+iv)} = e^u e^{iv}.$$

If $z = re^{i\theta_p}$; where $0 \leq \theta_p < 2\pi$ then we have:

$$re^{i\theta_p} = e^u e^{iv}.$$

So $r = e^u$, $r, u \in \mathbb{R}$ and $v = \theta_p + 2\pi n$, n an integer.

Since $r, u \in \mathbb{R}$ we know that $u = \ln(r)$.

By analogy we say that :

$$\boldsymbol{w = \ln(z) = \ln(r) + i(\theta_p + 2\pi n)}, \quad n \text{ any integer}, \ 0 \leq \theta_p < 2\pi.$$

Notice that $z = 0$ is a branch point for $w = \ln(z)$. As we go from $z = \epsilon$, ϵ a positive real number, around a circle of radius ϵ to $z = \epsilon e^{2\pi i}$, $w = \ln(z)$ goes from $\ln(\epsilon)$ to $\ln(\epsilon) + 2\pi i$. In fact, each time we go around the point $z = 0$ we get a different value for $\ln(z)$. Thus $\ln(z)$ has an infinite number of values ($w = z^{\frac{1}{2}}$ only has 2 values).

When we take $n = 0$, we get the principal value of the logarithm:

$$w = \ln(z) = \ln(r) + i(\theta_p), \quad 0 \leq \theta_p < 2\pi.$$

By restricting the value of θ_p to $0 \leq \theta_p < 2\pi$ we create a (continuous) single valued function. Thus we can again take as a branch cut for $ln(z)$, the positive real axis.

Notice that $z = \infty$ is also a branch point of $ln(z)$, since if we substitute $z = \frac{1}{t}$ then near $z = \infty$ and $t = 0$ we have:

$$\ln(z) = \ln\left(\frac{1}{t}\right) = -\ln(t).$$

Since $\ln(z)$ had a branch point at $z = 0$, $-\ln(t)$ will have a branch point at $t = 0$, which corresponds to $z = \infty$.

We will write $\mathbf{log}(z)$ **for** $\mathbf{ln}(z)$.

Now let's find $u(x, y)$ and $v(x, y)$ such that:

$$\mathrm{w} = \log(\mathrm{z}) = \mathrm{u}(\mathrm{x}, \mathrm{y}) + \mathrm{iv}(\mathrm{x}, \mathrm{y}).$$

We start with $z = e^w$; let $z = x + iy$, $w = u + iv$.

$$x + iy = e^{(u+iv)} = e^u e^{iv} = e^u(cosv + isinv)$$

So $\quad x = e^u cosv, \qquad y = e^u sinv.$

Thus: $\quad x^2 + y^2 = e^{2u}cos^2v + e^{2u}sin^2v = e^{2u}.$

$$\Rightarrow \qquad \log(x^2 + y^2) = 2u; \qquad \text{or} \qquad u(x, y) = \frac{1}{2}\log(x^2 + y^2).$$

$$\frac{y}{x} = \frac{e^u sinv}{e^u cosv} = tanv \qquad \text{or} \qquad v(x, y) = tan^{-1}(\frac{y}{x})$$

So:

$$\mathrm{w} = \log(\mathrm{z}) = \mathrm{u}(\mathrm{x}, \mathrm{y}) + \mathrm{iv}(\mathrm{x}, \mathrm{y}) = \frac{1}{2}\log(x^2 + y^2) + i\left(tan^{-1}\left(\frac{y}{x}\right)\right).$$

Note: Since $-\frac{\pi}{2} < \tan^{-1} A < \frac{\pi}{2}$, to guarantee that $v(x, y)$ is differentiable for $(x, y) \neq (0,0)$, when we write $v(x, y) = tan^{-1}\left(\frac{y}{x}\right)$ we really mean

$$v(x, y) = tan^{-1}\left(\frac{y}{x}\right) + k_j; \quad j = 1,2,3,4.$$

where j refers to the quadrant that (x, y) is in and

$$k_1 = 0, \quad k_2 = k_3 = \pi, \quad k_4 = 2\pi.$$

One can see the problem if you let (x, y) go around the unit circle. The value of $v(x, y)$ without the constants added will jump as (x, y) crosses the y-axis.

Let's show that $\mathrm{w} = \log(\mathrm{z})$ satisfies the Cauchy-Riemann equations when $z \neq 0$.

$$\log(\mathrm{z}) = \mathrm{u(x,y)} + \mathrm{iv(x,y)} = \frac{1}{2}\log(x^2 + y^2) + i\left(tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$u_x = \frac{x}{x^2+y^2} \qquad v_x = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$u_y = \frac{y}{x^2+y^2} \qquad v_y = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

So if $z \neq 0$, then $\qquad u_x = v_y \qquad u_y = -v_x$

and all partial derivatives are continuous away from $(0,0)$. Thus $w = \log(z)$ is analytic if $z \neq 0$.

To calculate $f'(z)$ when $f(z) = \log(z)$, recall that:

$$f'(z) = u_x + iv_x = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

Ex. Find all possible values of $Log(1+i)$ and identify its principal value.

We start by writing $1+i$ in polar exponential form.

$x = 1, \;\; y = 1$ so

$r = \sqrt{1^2+1^2} = \sqrt{2},$ and $tan\theta = \frac{y}{x} = \frac{1}{1} = 1.$

If you plot $1+i$ in the complex plane you can see that it's in the first quadrant and therefore, $\theta = \frac{\pi}{4}$. So we can write:

$$1+i = (\sqrt{2})e^{\left(\frac{\pi}{4}+2n\pi\right)i}, \quad \text{where} = 0, \pm 1, \pm 2, \dots .$$

$$Log(1+i) = Log(\sqrt{2}e^{\left(\frac{\pi}{4}+2n\pi\right)i})$$

$$= Log\sqrt{2} + \left(\frac{\pi}{4} + 2n\pi\right)i; \quad n = 0, \pm 1, \pm 2, \dots \; .$$

The principal value occurs when $n = 0$; so the principal value is:

$Log(1+i) = Log\sqrt{2} + \left(\frac{\pi i}{4}\right)$ or $Log(1+i) = \frac{1}{2}Log2 + \left(\frac{\pi i}{4}\right).$

Ex. Find all solutions of $3 - 2e^{2z+1} = 7$.

$$3 - 2e^{2z+1} = 7$$

$$e^{2z+1} = -2$$

$2z + 1 = \ln(-2);$ where $-2 = 2e^{(\pi i + 2n\pi i)}, \;\; n = 0, \pm 1, \pm 2, \dots$

$$2z + 1 = \ln\left(2e^{(\pi i+2n\pi i)}\right) = \ln(2) + \pi i(2n+1)$$

$$2z = \ln(2) - 1 + \pi i(2n+1)$$

$$z = \frac{(\ln(2)-1)}{2} + \frac{2n+1}{2}(\pi i); \quad n = 0, \pm 1, \pm 2, \dots \; .$$

We can now define z^t, where $z, t \in \mathbb{C}$ to be:

$$\boldsymbol{z^t = e^{t\log(z)}}.$$

Notice that if t is an integer we do not get a multiple valued function since if $t = k$, an integer then:

$$z^k = e^{k\log(z)} = e^{k(\log(r)+i(\theta_p+2\pi n))} = e^{klog(r)}e^{ik\theta_p}e^{2\pi nki}.$$

Since k and n are integers $e^{2\pi nki} = 1$.

So $\quad z^k = r^k e^{ik\theta_p}$ which is single valued.

If t is a rational number, $t = \frac{p}{q}$, p and q integers with no common factors and $q \neq 0$ then:

$$z^{\frac{p}{q}} = e^{\frac{p}{q}\log(z)} = e^{\frac{p}{q}log(r)}e^{i\frac{p}{q}\theta_p}e^{2\pi n\frac{p}{q}i} \quad ; \quad n = 0, \pm 1, \pm 2, \dots .$$

For $n = 1,2, \dots , q-1$; $\dfrac{np}{q}$ is not an integer and therefore $e^{2\pi n\frac{p}{q}i} \neq 1$.

Thus we get different values for $z^{\frac{p}{q}}$ for each value of $n = 0,1,2, \dots , q-1$. Thus $z^{\frac{p}{q}}$ is multivalued (in fact q-valued).

Once n get beyond $q-1$ or is less than 0, the values of $z^{\frac{p}{q}}$ will repeat. Thus $z^{\frac{p}{q}}$ has q branches.

We can now find $f'(z)$ for $f(z) = z^t$ by:

$$\frac{d}{dz}(z^t) = \frac{d}{dz}\left(e^{t\log(z)}\right) = \left(e^{t\log(z)}\right)\left(\frac{t}{z}\right) = z^t\left(\frac{t}{z}\right) = tz^{(t-1)}.$$

As we saw with $f(z) = z^{\frac{1}{2}}$, $f(z) = z^{\frac{p}{q}}$ has branch points at $z = 0 \; and \; \infty$ and a branch cut along the positive real axis will give us a single valued function. The principal value is again when $n = 0$:

$$f(z) = z^{\frac{p}{q}} = e^{\frac{p}{q}log(r)} e^{i\frac{p}{q}\theta_p} e^{2\pi n\frac{p}{q}i}$$

with $n = 0$ we get: $$f(z) = z^{\frac{p}{q}} = r^{\frac{p}{q}} e^{i\frac{p}{q}\theta_p} \; ; \qquad 0 \leq \theta_p < 2\pi.$$

Ex. Find all values of $(1 + i)^{\frac{2}{3}}$ and identify the principal value.

First write $1 + i$ in polar form.

$$z = 1 + i = \sqrt{2}\, e^{(\frac{\pi}{4}i + 2n\pi i)}; \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$z^{\frac{2}{3}} = \left(\sqrt{2}\right)^{\frac{2}{3}} e^{\left(\frac{\pi}{4}i + 2n\pi i\right)\frac{2}{3}} = (2^{\frac{1}{3}}) e^{(\frac{\pi}{6}i + \frac{4n\pi}{3}i)} .$$

$$n = 0 \qquad z^{\frac{2}{3}} = \sqrt[3]{2} e^{\frac{\pi}{6}i} = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{6}\right) + i sin\left(\frac{\pi}{6}\right)\right) = \sqrt[3]{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$$

$$n = 1 \qquad z^{\frac{2}{3}} = \sqrt[3]{2} e^{(\frac{\pi}{6}i + \frac{4\pi}{3}i)} = \sqrt[3]{2} e^{(\frac{3\pi}{2}i)} = -\sqrt[3]{2}i$$

$$n = 2 \qquad z^{\frac{2}{3}} = \sqrt[3]{2} e^{(\frac{\pi}{6}i + \frac{8\pi}{3}i)} = \sqrt[3]{2} e^{\frac{17\pi}{6}i} = \sqrt[3]{2} e^{\frac{5\pi}{6}i} = \sqrt[3]{2}(-\frac{\sqrt{3}}{2} + \frac{1}{2}i).$$

Values repeat for $n > 2$ or $n < 0$.

Principal Value= $\sqrt[3]{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$ (i.e. $n = 0$).

Ex. Let $f(z) = (z - z_0)^{\frac{1}{4}}$, find the branch points and possible branch cuts.

Let $z' = z - z_0$, then the function becomes $w = (z')^{\frac{1}{4}}$. We just saw that this function has branch points at $z' = 0$ and $z' = \infty$.

Thus $w = (z - z_0)^{\frac{1}{4}}$ has branch points at $0 = z' = z - z_0$ or $z = z_0$ and $\infty = z' = z - z_0$ or $z = \infty$.

We could use as a branch cut the ray starting at $z' = 0$ and going out the positive x' axis. This is the same as taking a ray starting at $0 = z' = z - z_0$ or a ray starting at $z = z_0$ and going parallel to the positive real axis. In fact, any ray starting at $z = z_0$ and going to ∞ will work as a branch cut.

<u>Inverse Trig Functions and Inverse Hyperbolic Functions</u>

Inverse trig functions and inverse hyperbolic functions are multivalued functions that can be calculated in terms of log functions.

$w = sin^{-1}(z)$ means that $\sin(w) = z$.

$$\frac{e^{iw} - e^{-iw}}{2i} = z$$

$$e^{iw} - e^{-iw} = 2iz$$

$$e^{2iw} - 1 = 2ize^{iw}$$

$$e^{2iw} - 2ize^{iw} - 1 = 0.$$

This is a quadratic equation in e^{iw} (you can substitute $s = e^{iw}$ and the equation becomes $s^2 - 2izs - 1 = 0$). We can solve it with the quadratic formula: $a = 1$, $b = -2iz$, $c = -1$.

$$e^{iw} = \frac{2iz+((2iz)^2-4(1)(-1))^{\frac{1}{2}}}{2}$$ (where $\frac{1}{2}$ power includes $\pm$ roots)

$$e^{iw} = \frac{2iz+(-4z^2+4)^{\frac{1}{2}}}{2} = iz + (1-z^2)^{\frac{1}{2}}$$ Now take logs on both sides:

$$iw = \log(iz + (1-z^2)^{\frac{1}{2}})$$

$$\boldsymbol{w = sin^{-1}(z) = -(i)\log(iz + (1 - z^2)^{\frac{1}{2}})}\textbf{).}$$

If you use the principal value of $(1-z^2)^{\frac{1}{2}}$, you get the principal value of the inverse sine.

We can now find a formula for the derivative of the inverse sine.

$$\frac{d}{dz}(sin^{-1}(z)) = \frac{d}{dz}(-(i)\log(iz + (1-z^2)^{\frac{1}{2}}))$$

$$= \left(\frac{-i}{iz+(1-z^2)^{\frac{1}{2}}}\right)\left(i - \frac{z}{(1-z^2)^{\frac{1}{2}}}\right)$$

$$= \left(\frac{-i}{iz+(1-z^2)^{\frac{1}{2}}}\right)\left(\frac{i(1-z^2)^{\frac{1}{2}}-z}{(1-z^2)^{\frac{1}{2}}}\right)$$

$$= (-i)\left(\frac{i}{(1-z^2)^{\frac{1}{2}}}\right) = \frac{1}{(1-z^2)^{\frac{1}{2}}}; \quad z \neq \pm 1.$$

We can similarly find that:

$$cos^{-1}(z) = -(i)\log(z + i(1-z^2)^{\frac{1}{2}}) \qquad \frac{d}{dz}\left(cos^{-1}(z)\right) = \frac{-1}{(1-z^2)^{\frac{1}{2}}}; \quad z \neq \pm 1$$

$$tan^{-1}(z) = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right) \qquad \frac{d}{dz}\left(tan^{-1}(z)\right) = \frac{1}{1+z^2}; \quad z \neq \pm i.$$

Ex. Derive formulas for $tanh^{-1}(z)$ and $\frac{d}{dz}(tanh^{-1}(z))$.

If $w = tanh^{-1}(z)$ then $\tanh(w) = z$.

So $\tanh(w) = \frac{\sinh(w)}{\cosh(w)} = \frac{e^w - e^{-w}}{e^w + e^{-w}} = z$

$$(\frac{e^w}{e^w})(\frac{e^w - e^{-w}}{e^w + e^{-w}}) = z$$

$$\frac{e^{2w} - 1}{e^{2w} + 1} = z$$

$$e^{2w} - 1 = z(e^{2w} + 1) = ze^{2w} + z$$

$$e^{2w} - ze^{2w} - 1 = z$$

$$e^{2w}(1 - z) = 1 + z$$

$$e^{2w} = \frac{1+z}{1-z}$$

$$2w = \log(\frac{1+z}{1-z})$$

$$\Rightarrow \qquad w = tanh^{-1}(z) = \frac{1}{2}\log(\frac{1+z}{1-z}) \, ; \qquad z \neq \pm 1.$$

The principal value of log gives the principal value of $w = tanh^{-1}(z)$.

$$\frac{d}{dz}(tanh^{-1}(z)) = \frac{d}{dz}(\frac{1}{2}\log(\frac{1+z}{1-z}))$$

$$= \frac{1}{2}[\frac{d}{dz}(\log(1+z) - \log(1-z))]$$

$$= \frac{1}{2}(\frac{1}{1+z} - \frac{1}{1-z}(-1))$$

$$= \frac{1}{2}(\frac{1}{1+z} + \frac{1}{1-z})$$

$$= \frac{1}{1-z^2}\ .$$

Similarly we can get:

$$sinh^{-1}(z) = \log(z + (1+z^2)^{\frac{1}{2}})\qquad \frac{d}{dz}\left(sinh^{-1}(z)\right) = \frac{1}{(1+z^2)^{\frac{1}{2}}}\,;\quad z \neq \pm i$$

$$cosh^{-1}(z) = \log(z + (z^2-1)^{\frac{1}{2}})\qquad \frac{d}{dz}\left(cosh^{-1}(z)\right) = \frac{1}{(z^2-1)^{\frac{1}{2}}}\,;\quad z \neq \pm 1.$$