The Cauchy-Riemann Equations

Theorem: A necessary (but not sufficient) condition for a function $f(z) = u(x, y) + iv(x, y)$ to be analytic (i.e. have a derivative) in a domain D is that u_x, u_y, v_x, v_y exist and satisfy the **Cauchy-Riemann equations**:

$$u_x = v_y\ , \qquad u_y = -v_x$$

at each point in D.

Note: If $f(z)$ is analytic at z_0 then we can calculate $f'(z_0)$ by:

$$f'(z_0) = u_x(x_0, y_0) + i\ v_x(x_0, y_0)$$

Proof:

Let $f(z) = u(x, y) + iv(x, y)$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(u(x,y) + iv(x,y)) - (u(x_0,y_0) + iv(x_0,y_0))}{z - z_0}.$$

This limit must exist, and be the same, along every path approaching z_0.

In particular it must work for:

Path 1: $y = y_0$, so $z - z_0 = x - x_0 = \Delta x$

Path 2: $x = x_0$, so $z - z_0 = (y - y_0)i = i\Delta y$.

Along Path 1 we have:

$$f'(z_0) = \lim_{x \to x_0} \frac{\big(u(x,y)+iv(x,y)\big)-(u(x_0,y_0)+iv(x_0,y_0))}{\Delta x}$$

$$= u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Along Path 2 we have:

$$f'(z_0) = \lim_{y \to y_0} \frac{\big(u(x,y)+iv(x,y)\big)-(u(x_0,y_0)+iv(x_0,y_0))}{i\Delta y}$$

$$= -iu_y(x_0, y_0) + v_y(x_0, y_0).$$

Since the limits have to be equal along both paths we have:

$$u_x(x_0, y_0) + iv_x(x_0, y_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0).$$

$$\Longrightarrow \quad u_x = v_y \,, \qquad u_y = -v_x \,.$$

Notice that this theorem says that you can't necessarily create a differentiable complex function $f(z) = u(x, y) + iv(x, y)$ by choosing any real valued functions $u(x, y)$ and $v(x, y)$ whose partial derivatives exist. It's not enough that the partial derivatives of $u(x, y)$ and $v(x, y)$ exist, they must also satisfy the **Cauchy-Riemann equations**: $u_x = v_y$, $u_y = -v_x$ just to have a chance for $f(z)$ to be analytic (even if the C-R equations are satisfied, that doesn't guarantee that the derivative of $f(z)$ exists).

Ex. Let $f(z) = u(x,y) + iv(x,y)$ if $z \neq 0$, and $f(z) = 0$ if $z = 0$, where

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2}; \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}; \quad u(0,0) = 0, \;\; v(0,0) = 0.$$

Show that the C-R equations are satisfied at $(0,0)$, but $f'(0)$ doesn't exist.

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x^3}{x^2}}{x} = 1$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y - 0} = \lim_{y \to 0} \frac{-\frac{y^3}{y^2}}{y} = -1$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x^3}{x^2}}{x} = 1$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y - 0} = \lim_{y \to 0} \frac{\frac{y^3}{y^2}}{y} = 1.$$

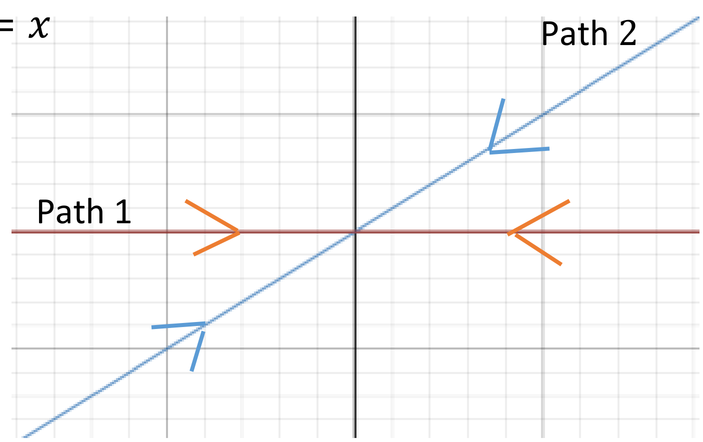
Thus we have: $u_x(0,0) = v_y(0,0) = 1; \quad u_y(0,0) = -\,v_x(0,0) = -1.$

So $u(x,y)$ and $v(x,y)$ satisfy the C-R equations.

Now let's calculate $\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$ by 2 paths to show that $f'(0)$ doesn't exist.

Path 1: let z approach 0 along the x axis (i.e. $y = 0$).

Path 2: let z approach 0 along the line $y = x$



Along Path 1: $\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{x\to 0}\frac{(u(x,0)+iv(x,0))-(u(0,0)+iv(0,0))}{x}$

$$= \lim_{x\to 0}\frac{\frac{x^3}{x^2}+i(\frac{x^3}{x^2})}{x} = 1 + i.$$

Along Path 2: $y = x$, so $u(x,x) = 0, \quad v(x,x) = x$

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{x\to 0}\frac{ix}{x+ix} = \frac{i}{1+i} = \frac{1}{2} + \frac{1}{2}i.$$

So $f'(0)$ doesn't exist even though $u(x,y)$ and $v(x,y)$ satisfy the C-R equations.

The "problem" is that u_x, u_y, v_x, v_y are not continuous at $(0,0)$.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$. A necessary and sufficient condition for f to be analytic in a domain D is that u_x, u_y, v_x, v_y exist, are continuous, and satisfy the Cauchy-Riemann equations, $u_x = v_y, \quad u_y = -v_x$.

Ex. Prove that the only complex analytic functions whose values (i.e. its range) are solely real numbers are constants.

If $f(z) = u(x,y) + iv(x,y)$ has only real values then $f(z) = u(x,y)$, since $v(x,y) = 0$.

If $f(z)$ is analytic then u and v must satisfy the C-R equations.

But $v(x,y) = 0 \implies v_x = 0$ and $v_y = 0$ at all points (x,y).

So the C-R equations then imply:

$$u_x = v_y = 0 \text{ and } u_y = -v_x = 0.$$

$\Rightarrow \quad u(x,y) =$constant since the partial derivatives of u are 0 everywhere.

Ex. Which of the following functions are analytic everywhere on $\mathbb{C}$ (i.e. are entire functions)?

a. $f(z) = z^2$
b. $f(z) = \bar{z}^2$
c. $f(z) = |z|^2$
d. $f(z) = e^z$

a. We already know $f(z) = z^2$ is analytic because we know how to take its derivative and we know that that exists everywhere. However, to check via our theorem:

$$f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy).$$

So $u(x, y) = x^2 - y^2$, $\quad v(x, y) = 2xy$

$u_x = 2x$, $\quad u_y = -2y$, $\quad v_x = 2y$, $\quad v_y = 2x$ (all continuous).

Thus we have: $u_x = v_y = 2x \quad u_y = -v_x = -2y$,

so $f(z) = z^2$ is analytic everywhere.

b. $f(z) = \bar{z}^2 = (x - iy)^2 = x^2 - y^2 - 2ixy.$

So $u(x, y) = x^2 - y^2$, $\quad v(x, y) = -2xy$

$u_x = 2x$, $\quad u_y = -2y$, $\quad v_x = -2y$, $\quad v_y = -2x$.

$$u_x = v_y \Longrightarrow 2x = -2x; \qquad u_y = -v_x \Rightarrow -2y = 2y.$$

But the only point that satisfies both C-R equations is $(0,0)$. However, a function can't be analytic only at a single point (to be analytic at a point it must be analytic in a neighborhood of that point). Thus this function is not analytic anywhere.

c. $f(z) = |z|^2 = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$.

So $u(x, y) = x^2 + y^2,\ v(x, y) = 0$ and we already saw that the only analytic function whose values are all real is a constant function. Thus $f(z) = |z|^2$ is not analytic anywhere.

d. $f(z) = e^z = e^{(x+iy)} = e^x(cosy + isiny)$

$$u(x, y) = e^x cosy, \qquad v(x, y) = e^x siny$$

$$u_x = e^x cosy, \quad u_y = -e^x siny, \qquad v_x = e^x siny, \quad v_y = e^x cosy$$

So all partials are continuous.

$$u_x = v_y = e^x cosy, \quad u_y = -v_x = -e^x siny.$$

Thus $f(z) = e^z$ is analytic everywhere on $\mathbb{C}$.

Def. A point z_0 is called an **isolated singularity** of $f(z)$ if $f(z)$ is not analytic at z_0 but is analytic in a deleted neighborhood of z_0 (a deleted neighborhood of z_0 is a neighborhood of z_0 with the point z_0 excluded).

Ex. $f(z) = \frac{1}{z-i}$

has an isolated singularity at $z = i$.

We know from our derivative formulas that $f'(z)$ exists for $z \neq i$. At $z = i$ the function is not defined so it can't have a derivative at $z = i$.

Laplace's Equation

Laplace's (differential) equation comes up frequently in physics $(2$ dimensional ideal fluid flow, steady state heat conduction, electrostatics, etc.). In 2 dimensions the equation is:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad \text{or} \qquad u_{xx}(x,y) + u_{yy}(x,y) = 0.$$

This is sometimes written as: $\nabla^2 u = 0$ or $\Delta u = 0$.

Def. Any real valued function $u(x,y)$ that is a solution to Laplace's equation and has continuous second partial derivatives is called **harmonic**.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in $D \subseteq \mathbb{C}$, then $u(x,y)$ and $v(x,y)$ are harmonic in D.

Proof: Since $f(z)$ is analytic in D, $u(x,y)$ and $v(x,y)$ must satisfy the C-R equations: $u_x = v_y \quad u_y = -v_x$.

We will see later that if $f'(z)$ exists in D (i.e. $f(z)$ is analytic in D) then $f''(z)$, $f'''(z)$, ..., $f^{(n)}(z)$, ... also exist, which implies that $u(x,y)$ and $v(x,y)$ have an infinite number of partial derivatives.

Since $u_x = v_y$ then $u_{xx} = v_{yx}$

$u_y = -v_x$ then $u_{yy} = -v_{xy}$.

But since $v_{xy} = v_{yx}$, we have $u_{xx} = -u_{yy}$ or $u_{xx} + u_{yy} = 0$.

Thus $u(x,y)$ is harmonic.

Since $u_x = v_y$ then $u_{xy} = v_{yy}$

$u_y = -v_x$ then $u_{yx} = -v_{xx}$.

But since $u_{xy} = u_{yx}$, we have $v_{yy} = -v_{xx}$ or $v_{xx} + v_{yy} = 0$.

Thus $v(x, y)$ is harmonic.

Def. If $u(x, y)$ and $v(x, y)$ are harmonic functions in a domain D such that

$f(z) = u(x, y) + iv(x, y)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ are called **harmonic conjugates**.

Ex. let $u(x, y) = x^3 - 3xy^2$. Show that $u(x, y)$ is harmonic for all real x, y, and find all harmonic conjugates and determine the corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

$u_x = 3x^2 - 3y^2$ $\qquad$ $u_y = -6xy$

$u_{xx} = 6x$ $\qquad$ $u_{yy} = -6x$ (partial derivatives are continuous)

So $u_{xx} + u_{yy} = 6x - 6x = 0$, thus $u(x, y)$ is harmonic.

Now we must find a $v(x, y)$ such that:

$u_x = 3x^2 - 3y^2 = v_y$

$u_y = -6xy = -v_x$

i.e. $u(x, y)$ and $v(x, y)$ must satisfy the C-R equations.

Finding $v(x,y)$ is a lot like finding a potential function for a gradient vector field. We start by integrating either equation with respect to the relevant variable.

$v_x = 6xy$ so $v(x,y) = \int (6xy)\, dx = 3x^2y + g(y)$.

Now differentiate $v(x,y)$ with respect to y.

$v_y = 3x^2 + g'(y) = 3x^2 - 3y^2$.

Thus $g'(y) = -3y^2$ and $g(y) = -y^3 + C$.

Hence $v(x,y) = 3x^2y - y^3 + C$ gives us all of the harmonic conjugates for $u(x,y) = x^3 - 3xy^2$.

This means that:

$$f(z) = u(x,y) + iv(x,y) = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

is analytic.

$$f(z) = u(x,y) + iv(x,y) = (x^3 - 3xy^2) + i(3x^2y - y^3 + C)$$

is also analytic for any constant C.

In order to find an expression for $f(z)$ in terms of z (not x and y) we need to guess at what this might be based on $u(x,y)$ and $v(x,y)$.

Notice that $z^3 = (x+iy)^3 = x^3 + 3x^2yi - 3xy^2 - iy^3$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

So $f(z) = z^3$. In general $f(z) = z^3 + C$.