The Derivative of a Complex Function

Def. Let $w = f(z)$ be a single valued function defined in a domain $D \subseteq \mathbb{C}$. We define the derivative of $f(z)$ at $z_0 \in D$, $f'(z_0)$ as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

if this limit exists.

Alternatively, we could define $f'(z_0)$ as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

if this limit exists. However, note that h can be thought of as any sequence of complex numbers approaching 0.

Def. A single valued function is said to be **analytic**, or **holomorphic**, or **regular**, in a domain D if it has a derivative at every point in D.

Def. A function is said to be **analytic (or holomorphic, or regular) at a point** $\boldsymbol{z_0 \in D}$, if $f(z)$ is analytic in any neighborhood of z_0**.**

Def. A function that is anayltic/holomorphic/regular at every point $z \in \mathbb{C}$ is called an **entire** function.

As with real valued functions:

Theorem: If $f(z)$ has a derivative at a point z_0, then $f(z)$ is continuous at z_0.

Proof: $\lim_{z\to z_0}(f(z) - f(z_0)) = \lim_{z\to z_0}[\left(\frac{f(z)-f(z_0)}{z-z_0}\right)(z - z_0)]$

$$= f'(z_0) \lim_{z\to z_0}(z - z_0)]$$

$$= 0.$$

$$\Longrightarrow \qquad \lim_{z\to z_0} f(z) = f(z_0).$$

So $f(z)$ is continuous at z_0.

Thus differentiable at z_0 implies continuous at z_0, but continuous at z_0 does not imply differentiable at z_0.

Ex. Prove that $f(z) = \bar{z}$ is continuous everywhere, but not differentiable anywhere.

To prove $f(z) = \bar{z}$ is continuous everywhere we must show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|z - z_0| < \delta$ then $|\bar{z} - \bar{z}_0| < \epsilon$.

Since $|w| = |\overline{w}|$

$$|z - z_0| = |\overline{z - z_0}| = |\bar{z} - \overline{z_0}|, \text{ so}$$

choose $\delta = \epsilon$.

Then $|z - z_0| < \delta = \epsilon$.

But $|z - z_0| = |\bar{z} - \overline{z_0}|$ so $|\bar{z} - \bar{z}_0| < \epsilon$.

Thus $f(z) = \bar{z}$ is continuous everywhere.

Now let's show that $f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ doesn't exist for any z_0.

$$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h \to 0} \frac{\overline{z_0+h}-\overline{z_0}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

If we let h approach 0 along the real axis, that is $h = x,\ x \in \mathbb{R}$, then $\bar{h} = h$.

Thus $\lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h \to 0} \frac{h}{h} = 1$ along that path.

If we let h approach 0 along the imaginary axis, that is $h = yi,\ y \in \mathbb{R}$, then $\bar{h} = -h$.

Thus $\lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$ along that path.

Hence $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ doesn't exist because we get different values with different paths as h goes to 0.

Thus $f(z) = \bar{z}$ is not differentiable anywhere.

Theorem: Let $f(z) = c, \quad g(z) = z,$ then

a. $f'(z) = 0$
b. $g'(z) = 1$

This follows directly from the definition of a derivative.

As with functions of one real variable we have the following differentiation rules:

Theorem: Let $w = f(z)$ and $w = g(z)$ be differentiable at $z \in \mathbb{C}$, then

1. $\big(f(z) \pm g(z)\big)' = f'(z) \pm g'(z)$
2. $\big(f(z)g(z)\big)' = f(z)g'(z) + g(z)f'(z)$ (product rule)
3. $\left(\frac{f(z)}{g(z)}\right)' = \frac{g(z)f'(z) - f(z)g'(z)}{\left(g(z)\right)^2}$ (quotient rule).

Theorem (chain rule): If $f(z)$ is differentiable in a neighborhood of $z_0 \in D$, and $g(z)$ is differentiable in a neighborhood of $f(z_0)$ and $w = g(f(z))$ then

$$\frac{dw}{dz} = g'\big(f(z_0)\big)f'(z_0).$$

Ex. If $w = (2z^3 + z + 1)^{10}$ then $\frac{dw}{dz} = 10(2z^3 + z + 1)^9(6z^2 + 1)$.

Theorem: if $f(z) = e^z$, then $f'(z) = e^z$.

Proof: $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

$$= \lim_{h \to 0} \frac{e^{(z+h)} - e^z}{h}$$

$$= \lim_{h \to 0} \frac{e^z(e^h - 1)}{h} \quad .$$

$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots;$ so

$e^h - 1 = h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots = h(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots)$

$$f'(z) = \lim_{h\to 0} \frac{e^z(e^h-1)}{h}$$

$$= e^z \lim_{h\to 0} \frac{h(1+\frac{h}{2!}+\frac{h^2}{3!}+\cdots)}{h}$$

$$= e^z.$$

Note: We are "cheating" a bit here because we don't know yet that the above power series calculations are "legitimate".

Since trig functions and hyperbolic functions are defined in terms of the function e^z, we can now find their derivatives using the derivative rules.

Theorem: $(\sin(z))' = cosz,$ $(\cos(z))' = -sinz.$

Proof: $$\frac{d}{dz}(sinz) = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)$$

$$= \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right)$$

$$= \frac{e^{iz}+e^{-iz}}{2} = cosz.$$

$$\frac{d}{dz}(cosz) = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right)$$

$$= \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right)$$

$$= -\frac{e^{iz}-e^{-iz}}{2i} = -sinz .$$

Since $(\sin(z))' = cosz$, $(\cos(z))' = -sinz$, we can get the derivatives of the other trig functions using the derivative rules:

$\frac{d}{dz}(tanz) = sec^2 z$, $\frac{d}{dz}(cscz) = -(cscz)(cotz)$,

$\frac{d}{dz}(cotz) = -csc^2 z$ $\frac{d}{dz}(secz) = (secz)(tanz)$.

Given that $sinh(z) = \frac{e^z - e^{-z}}{2}$ and $cosh(z) = \frac{e^z + e^{-z}}{2}$:

$$\frac{d}{dz}(\sinh(z)) = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^z + e^{-z}}{2} = \cosh(z)$$

$$\frac{d}{dz}(\cosh(z)) = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^z - e^{-z}}{2} = \sinh(z).$$

We can now use the derivative rules to find:

$\frac{d}{dz}(\tanh(z)) = sech^2(z)$ $\frac{d}{dz}(\text{csch}(z)) = -(\text{csch}(z))(\coth(z))$

$\frac{d}{dz}(\coth(z)) = -csch^2(z)$ $\frac{d}{dz}(\text{sech}(z)) = -(\text{sech}(z))(\tanh(z))$.

Ex. Where are the following functions differentiable?

a. $\frac{z+2}{z^2+9}$ b. $secz$

a. The derivative of $R(z) = \frac{P(z)}{Q(z)}$, is defined for all z such that $Q(z) \neq 0$.

In this case, where $z^2 + 9 \neq 0$.

$z^2 + 9 = 0$

$(z + 3i)(z - 3i) = 0$

$z = \pm 3i$

So $\frac{z+2}{z^2+9}$ is differentiable for all $z \in \mathbb{C}$ such that $z \neq \pm 3i$.

b. $secz = \frac{1}{cosz} = \frac{1}{\frac{e^{iz}+e^{-iz}}{2}} = \frac{2}{e^{iz}+e^{-iz}}$

So the domain of $secz$ is all $z \in \mathbb{C}$ such that $e^{iz} + e^{-iz} \neq 0$.

$e^{iz} + e^{-iz} = 0$

$e^{2iz} + 1 = 0$

$e^{2iz} = -1 = e^{(\pi i + 2\pi n i)}, \qquad n = 0, \pm 1, \pm 2, \ldots$

$2iz = (2n + 1)\pi i$

$z = \frac{(2n+1)}{2}\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$

So the domain of $secz$ is all $z \in \mathbb{C}$ such that

$z \neq \frac{(2n+1)}{2}\pi, \;\; n = 0, \pm 1, \pm 2, \ldots$.

We also know that $\frac{d}{dz}(secz) = (secz)(tanz)$. This is defined for all z such that $cosz \neq 0$ (the same domain as $secz$). So the derivative of $secz$ exists for all $z \in \mathbb{C}$ such that $z \neq \frac{(2n+1)}{2}\pi, \;\; n = 0, \pm 1, \pm 2, \ldots$.