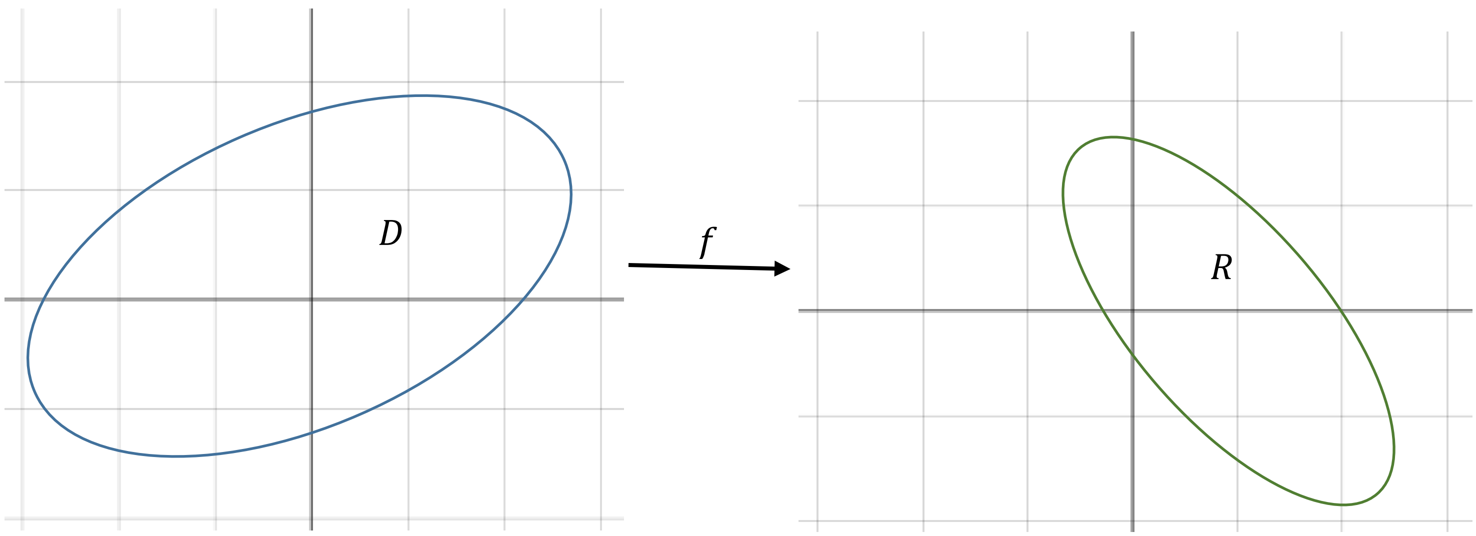
Limits and Continuity

Def. We say $w = f(z)$ is a **complex valued function** (or a complex function) of a complex variable z with a **domain** D and **range** R if D and R are nonempty subsets of $\mathbb{C}$, and for each point $z \in D$ there corresponds at least one point $w \in R$, and for each point $w \in R$ there is at least one point $z \in D$.



If no two points of R correspond to the same $z \in D$, we say $f(z)$ is **single-valued.**

Notice that unlike functions defined in elementary math, we are allowing a point in the domain to get mapped to 2 or more points in the range. If this happens we say that $f(z)$ is a **multi-valued function.**

A complex function $w = f(z)$ can also be written in the form:

$$w = f(z) = u(x, y) + iv(x, y)$$

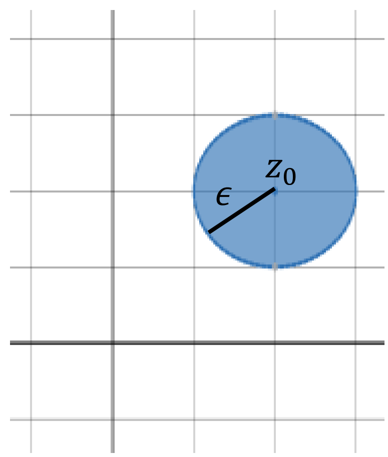
where $u(x, y)$ and $v(x, y)$ are real valued functions of real variables.

Ex. Write $f(z) = z^2 + 2z + 3$ as $f(z) = u(x, y) + iv(x, y)$.

$$\begin{aligned} f(z) &= (x + iy)^2 + 2(x + iy) + 3 \\ &= (x^2 + 2xyi + i^2y^2) + (2x + 2yi) + 3 \\ &= (x^2 - y^2 + 2xyi) + (2x + 2yi) + 3 \\ &= (x^2 - y^2 + 2x + 3) + (2xy + 2y)i. \end{aligned}$$

In this case: $u(x, y) = x^2 - y^2 + 2x + 3$ and $v(x, y) = 2xy + 2y$.

Def. A **neighborhood** of a point z_0 in the complex plane is the set of points z such that $|z - z_0| < \epsilon,$ where ϵ is a positive real number.



If $z = x + yi$ and $z_0 = x_0 + y_0 i$ then:

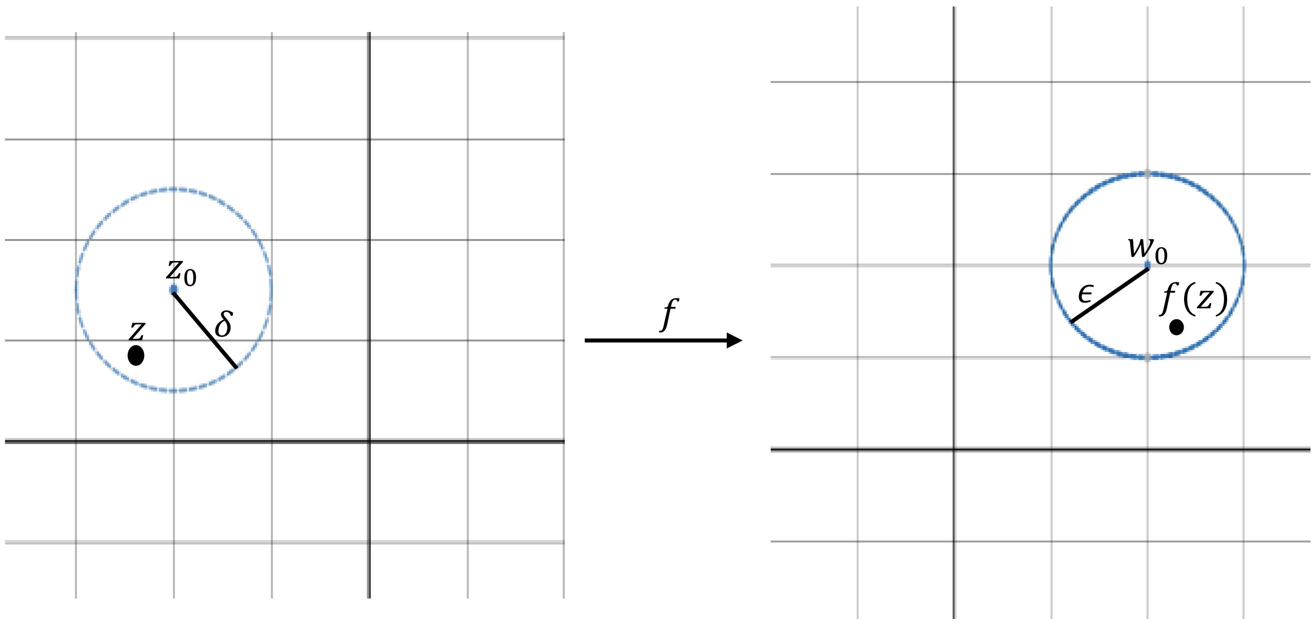
$$\begin{aligned} |z - z_0| &= |(x + yi) - (x_0 + y_0 i)| \\ &= |(x - x_0) + (y - y_0)i| \\ &= \sqrt{(x - x_0)^2 + (y - y_0)^2} \end{aligned}$$

So $|z - z_0| < \epsilon$ means

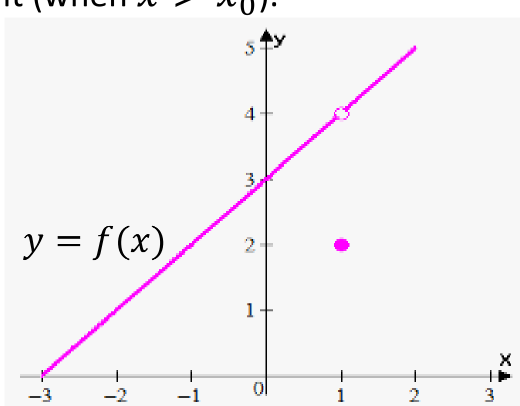
$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon \quad \text{or} \quad (x - x_0)^2 + (y - y_0)^2 < \epsilon^2$$

Which represents all of the points inside (but not including) the circle in the xy-plane whose center is (x_0, y_0) and whose radius is ϵ.

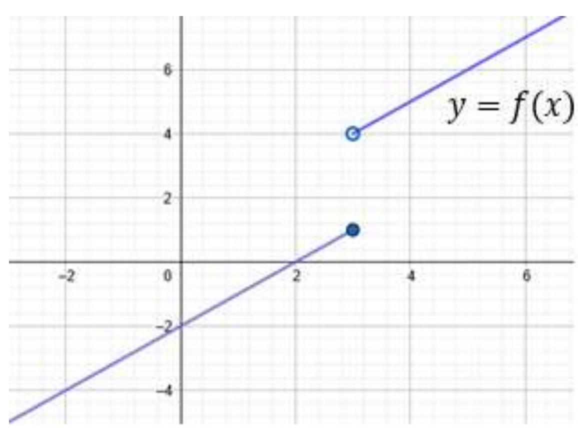
Def. Let $w = f(z)$ be a single valued function defined in a domain D except possibly at the point $z_0 \in D$. We say $\lim\limits_{z \to z_0} f(z) = w_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |z - z_0| < \delta$ then $|f(z) - w_0| < \epsilon$.



As with real valued functions, the function $f(z)$ need not be defined at $z = z_0$. For a function of one real variable, in order for $\lim\limits_{x \to x_0} f(x) = L$, we need the value of the function f to approach L from the left (when $x < x_0$) and from the right (when $x > x_0$).

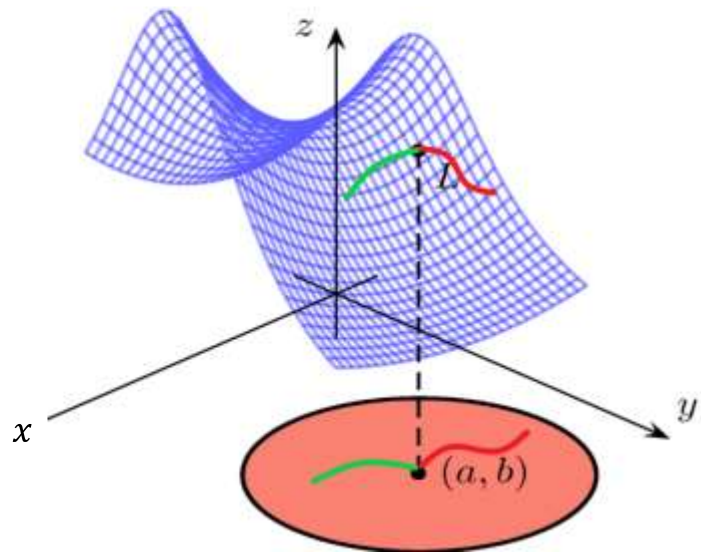


The $\lim\limits_{x \to 1} f(x)$ exists.

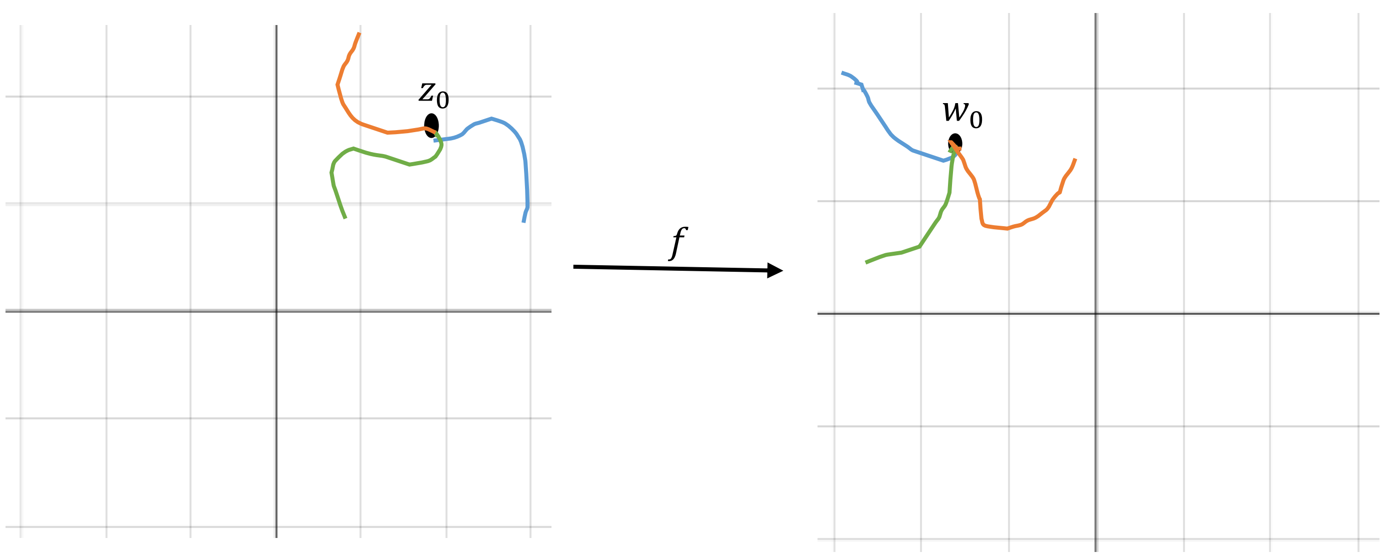


The $\lim\limits_{x \to 3} f(x)$ doesn't.

For a real valued function of two real variables, for $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we needed the limit from every direction to exist and be equal to L as (x,y) approches (a,b).



For a function of one complex variable, for $\lim_{z\to z_0} f(z) = w_0$, we also need the limit to exist and be equal to w_0, as z approaches z_0 from all directions.



Def. $\lim_{z\to\infty} f(z) = w_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|z| > \frac{1}{\delta}$ then $|f(z) - w_0| < \epsilon$. (This is the same as $\lim_{w\to 0} f\left(\frac{1}{w}\right) = w_0$.)

Ex. Show that $\lim_{z\to\infty} \frac{1}{z} = 0$.

Start with the epsilon statement: $\left|\frac{1}{z} - 0\right| < \epsilon \Leftrightarrow \ |z| > \frac{1}{\epsilon}$.

Choose $\delta = \epsilon$

So $|z| > \frac{1}{\delta} = \frac{1}{\epsilon}$

$\left|\frac{1}{z}\right| < \epsilon \Rightarrow \left|\frac{1}{z} - 0\right| < \epsilon.$

Thus $\lim_{z\to\infty} \frac{1}{z} = 0.$

All of the standard properties of limits for functions of a real variable hold for $w = f(z)$.

Theorem: If $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$ then

1. $\lim_{z\to z_0} \big(f(z) + g(z)\big) = w_0 + w_1$
2. $\lim_{z\to z_0} \big(f(z) - g(z)\big) = w_0 - w_1$
3. $\lim_{z\to z_0} \big(f(z)g(z)\big) = w_0\, w_1$
4. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$; as long as $w_1 \neq 0$.

It's easy to show with a δ/ϵ proof that $\lim_{z \to z_0}(az+b) = az_0 + b$ for all $a, b, z_0 \in \mathbb{C}$. Thus it follows from the previous theorem that:
$\lim_{z \to z_0}(a_0 + a_1 z + \cdots + a_n z^n) = a_0 + a_1 z_0 + \cdots + a_n (z_0)^n$. So for any polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$; $\lim_{z \to z_0} P(z) = P(z_0)$.

It also follows from the limit theorem that for any rational function $R(z) = \frac{P(z)}{Q(z)}$ (where $P(z)$ and $Q(z)$ are polynomials) $\lim_{z \to z_0} R(z) = R(z_0)$ as long as $Q(z_0) \neq 0$.

Thus for polynomials and rational functions where $Q(z_0) \neq 0$, we can evaluate limits by just plugging in the point z_0 into the function.

Ex. Evaluate $\lim_{z \to i+1} \frac{z^2}{z-1}$.

$$\lim_{z \to i+1} \frac{z^2}{z-1} = \frac{(i+1)^2}{(i+1)-1} = \frac{1+2i+i^2}{i} = \frac{2i}{i} = 2.$$

Ex. Evaluate $\lim_{z \to \infty} \frac{z^2}{1-2z^2}$.

To evaluate $\lim_{z \to \infty} \frac{z^2}{1-2z^2}$, make a substitution $z = \frac{1}{w}$ and take the limit as w→ 0.

$$\lim_{z \to \infty} \frac{z^2}{1-2z^2} = \lim_{w \to 0} \frac{(\frac{1}{w})^2}{1-2(\frac{1}{w})^2} = \lim_{w \to 0} \frac{\frac{1}{w^2}}{1-\frac{2}{w^2}}$$

$$= \lim_{w \to 0} \frac{\frac{1}{w^2}}{1-\frac{2}{w^2}}\left(\frac{w^2}{w^2}\right) = \lim_{w \to 0} \frac{1}{w^2-2} = -\frac{1}{2}.$$

<u>Continuity</u>

Def. Let $w = f(z)$ be defined in a domain $D \subseteq \mathbb{C}$. We say $\boldsymbol{f(z)}$ **is continuous at** $\boldsymbol{z_0} \in \boldsymbol{D}$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

As with real valued functions, in order for a function $w = f(z)$ to be continuous at $z = z_0$, three things must hold:

1. $f(z_0)$ must be defined
2. $\lim_{z \to z_0} f(z)$ must exist
3. $\lim_{z \to z_0} f(z) = f(z_0)$

If $f(z)$ is continuous at every point $z \in D \subseteq \mathbb{C}$, we say $f(z)$ is **continuous on** $\boldsymbol{D}$.

As with real valued functions, the following theorem follows from our earlier limit theorem.

Theorem: If $f(z)$ and $g(z)$ are continuous at $z_0 \in D$ then

1. $f(z) + g(z)$
2. $f(z) - g(z)$
3. $f(z)g(z)$
4. $\frac{f(z)}{g(z)}, \quad g(z_0) \neq 0$

   are all continuous at z_0.

It also follows that polynomials, $P(z)$, and rational functions, $R(z) = \frac{P(z)}{Q(z)}$, $Q(z_0) \neq 0$, are continuous at any point $z_0 \in D$.