Elementary Functions

For $x \in \mathbb{R}$, we can define $y = e^x$ by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

From one variable Calculus we know that this converges for all $x \in \mathbb{R}$.

We can extend this definition to e^{ix}, $x \in \mathbb{R}$ by

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots + \frac{(ix)^n}{n!} + \cdots .$$

This definition is not "legitimate" because we haven't talked about convergence of a series in $\mathbb{C}$ yet (but we will), but we will use this definition for now.

We saw earlier that:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = cosx + isinx.$$

In order to define e^z, for $z \in \mathbb{C}$ we want to make sure that the usual exponent rules still hold: $e^{(a+b)} = e^a \cdot e^b, etc.$. This leads us to define e^z by:

$$e^z = e^{(x+iy)} = e^x e^{iy} = e^x(cosy + isiny).$$

This definition agrees with the definition of e^x when $x \in \mathbb{R}$, (i.e. when $y = 0$) and e^{iy}, $y \in \mathbb{R}$, (i.e. when $x = 0$).

Notice that with this definition:

1. $e^z \neq 0$ for any $z \in \mathbb{C}$
2. $|e^z| = |e^{(x+iy)}| = e^x > 0$, since $|e^{iy}| = 1$
3. $e^{\frac{\pi i}{2}} = i$, $e^{\pi i} = -1$, $e^{\frac{3\pi i}{2}} = -i$, $e^{2\pi i} = 1$
4. $e^z = 1$ if and only if $z = 2\pi i n$, where n is an integer.

Proof: 1&2,

$$e^z = e^{(x+iy)} = e^x e^{iy} = e^x(cosy + isiny) \text{ thus}$$

$$|e^z|^2 = (e^z)(\overline{e^z}) = (\,e^x(cosy + isiny))(\,e^x(cosy - isiny))$$

$$= e^{2x}\left(cos^2 y + sin^2 y\right) = e^{2x} > 0.$$

So we have:

$$|e^z| = |e^{(x+iy)}| = e^x > 0 \text{ and } e^z \neq 0 \text{ for any } z \in \mathbb{C}.$$

For 3 just plug into the formula: $e^{ix} = cosx + isinx$.

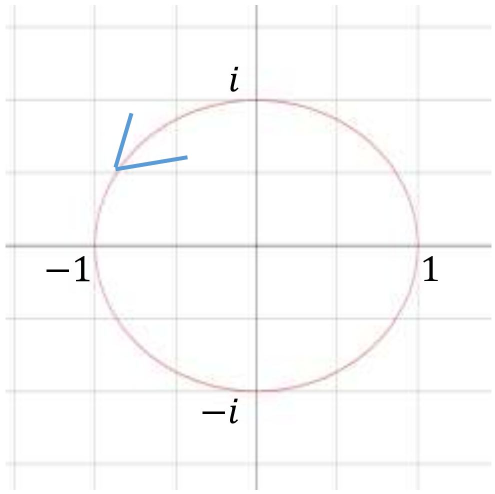
For 4: $e^z = e^x(cosy + isiny) = 1 \Rightarrow siny = 0$ and $e^x cosy = 1$.

But $siny = 0$ means $cosy = \pm 1$.

Since $e^x > 0$ for $x \in \mathbb{R}$,
$e^x cosy = 1$ implies that $cosy = 1$ and $e^x = 1$.

Thus $x = 0$, and $y = 2\pi n$, where n is an integer.

Since if $x \in \mathbb{R}$, $|e^{ix}| = |cosx + isinx| = 1$, it means that e^{ix} lies on the unit circle for all real values of x. At $x = 0$, $e^{ix} = e^0 = 1$. As x moves from 0 to 2π, e^{ix} moves counterclockwise around the unit circle.



Definitions of Trig functions of complex numbers

We know that:

$e^{ix} = cosx + isinx$, and $e^{-ix} = cosx - isinx$, thus by adding the equations and dividing by 2 we get:

$$cosx = \frac{e^{ix}+e^{-ix}}{2}.$$

If we subtract the 2 equations and divide by $2i$ we get:

$$sinx = \frac{e^{ix}-e^{-ix}}{2i}.$$

This gives us a natural way to define $sinz$ and $cosz$ for $z \in \mathbb{C}$:

$$cosz = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad sinz = \frac{e^{iz}-e^{-iz}}{2i}.$$

It's easy to check that the usual trig relationships hold with this definition. For example:

$$sin^2 z + cos^2 z = (\frac{e^{iz} - e^{-iz}}{2i})^2 + (\frac{e^{iz} + e^{-iz}}{2})^2$$
$$= \left(\frac{e^{2iz} - 2 + e^{-2iz}}{-4}\right) + (\frac{e^{2iz} + 2 + e^{-2iz}}{4})$$
$$= 1.$$

Ex. Find the values of $\sin(i)$ and $\cos(i)$.

$$\sin(i) = \frac{e^{i(i)} - e^{-i(i)}}{2i} = \frac{e^{i^2} - e^{-(i)^2}}{2i}$$
$$= \frac{e^{-1} - e^{1}}{2i} = -i\,\frac{(e^{-1} - e)}{2}$$ (which is purely imaginary).

$$\cos(i) = \frac{e^{i(i)} + e^{-i(i)}}{2} = \frac{e^{i^2} + e^{-(i)^2}}{2}$$
$$= \frac{e^{-1} + e^{1}}{2} = \frac{(e^{-1} + e)}{2}$$ (which is real).

Note: A very familiar inequalitiy concerning the value of the sine and cosine of a real number, namely:

$|sinx| \leq 1$ and $|cosx| \leq 1$ for all real numbers x

does NOT hold for the sine and cosine of complex numbers. For example:

$$|\cos(-100i)| = \left|\frac{e^{i(-100i)} + e^{i(100i)}}{2}\right| = \left|\frac{e^{100} + e^{-100}}{2}\right|$$ which is a very large.

$$|\sin(-100i)| = \left|\frac{e^{i(-100i)} - e^{i(100i)}}{2i}\right| = \left|\frac{e^{100} - e^{-100}}{2}\right|$$ is also a very large.

We can now define the other four trig functions in terms of $sinz$ and $cosz$.

$$tanz = \frac{sinz}{cosz}; \qquad cscz = \frac{1}{sinz}; \qquad secz = \frac{1}{cosz}; \qquad cotz = \frac{cosz}{sinz} = \frac{1}{tanz}.$$

<u>Definitions of Hyperbolic Functions</u>

For all real numbers $x \in \mathbb{R}$ we have:

$$sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$cosh(x) = \frac{e^x + e^{-x}}{2} \; .$$

We can now extend these definitions to $z \in \mathbb{C}$ by

$$sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$cosh(z) = \frac{e^z + e^{-z}}{2} \; .$$

Again the usual identities hold for all $z \in \mathbb{C}$. For example:

$$cosh^2(z) - sinh^2(z) = 1$$

$$\left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \left(\frac{e^{2z} + 2 + e^{-2z}}{4}\right) - \left(\frac{e^{2z} - 2 + e^{-2z}}{4}\right) = 1.$$

We define the other four hyperbolic functions from $sinh(z)$ and $cosh(z)$:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} \qquad \operatorname{csch}(z) = \frac{1}{\sinh(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} \qquad \operatorname{sech}(z) = \frac{1}{\cosh(z)} \;.$$

There are relationships between trig functions and hyperbolic functions.

Theorem: $\sin(ix) = (i)\sinh(x)$ for $x \in \mathbb{R}$

$\cos(ix) = \cosh(x)$ for $x \in \mathbb{R}$.

Proof: $$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i\left(\frac{e^{x} - e^{-x}}{2}\right) = (i)\sinh(x)$$

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh(x).$$

<u>Power Series Representations</u>

Given a power series representation for a function $f(z)$ about a point $z = z_0$:

$$f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j\,; \qquad a_j, z_0 \in \mathbb{C}$$

we can ask where the power series converges. We establish for what values of z a power series converges via the ratio test (as was done in single variable Calculus). The power series converges for all values of z where:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||z - z_0| < 1.$$

That is, it converges inside the circle $|z - z_0| = R$, where

$$R = \text{Radius of convergence} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

We can derive power series expansions for $sinz$, $cosz$, $\sinh(z)$, and $\cosh(z)$ from the power series for e^z.

Ex. Find a power series expansion for $f(z) = cosz$ and determine for which values of z it converges. Based on this, where does the power series for $cos^4 z$ and $\sec(z)$ converge.

$$f(z) = cosz = \frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left[\sum_{j=0}^{\infty}\frac{(iz)^j}{j!} + \sum_{j=0}^{\infty}\frac{(-iz)^j}{j!}\right]$$

$$= \frac{1}{2}\sum_{j=0}^{\infty}\frac{((i)^j+(-i)^j)z^j}{j!} = \frac{1}{2}\sum_{j=0}^{\infty}\frac{(1+(-1)^j)i^j z^j}{j!}$$

$1 + (-1)^j = 0$ if j is odd

$= 2$ if j is even.

$$\Longrightarrow \quad cosz = \sum_{j=0}^{\infty}\frac{(-1)^j z^{2j}}{(2j)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots.$$

Radius of convergence = $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$.

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n}{(2n)!}\frac{(2n+2)!}{(-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty}|(2n+2)(2n+1)| = \infty.$$

So the radius of convergence is ∞, thus the power series for $f(z) = cosz$ converges for all $z \in \mathbb{C}$.

Notice that the power series for $cosz$ looks very similar to the power series for $cosx$, when x is a real number.

Since the power series for $f(z) = cosz$ converges for all $z \in \mathbb{C}$, so does the power series for $g(z) = cos^4 z$, since you can get that series by multiplying the series for $cosz$ by itself 4 times.

The power series for $h(z) = secz = \frac{1}{cosz}$ converges everywhere the series for $cosz$ does except where $cosz = 0$. Thus the power series for $secz$ converges for all $z \in \mathbb{C}$ except when $z = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$.

Ex. Find a power series representation for $f(z) = \frac{cosz-1}{z^2}$.

$$cosz = \sum_{j=0}^{\infty}\frac{(-1)^j z^{2j}}{(2j)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + \frac{(-1)^j z^{2j}}{(2j)!} + \cdots$$

$$\frac{cosz-1}{z^2} = \frac{-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots+\frac{(-1)^j z^{2j}}{(2j)!}+\cdots}{z^2}$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots + \frac{(-1)^j z^{2j-2}}{(2j)!} + \cdots = \sum_{j=1}^{\infty}\frac{(-1)^j z^{(2j-2)}}{(2j)!}.$$

Ex. Find a power series for $\sinh(z)$ and determine its radius of convergence.

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right]$$

$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) - \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[2z + \frac{2z^3}{3!} + \frac{2z^5}{5!} + \frac{2z^7}{7!} + \cdots + \frac{2z^{2j+1}}{(2j+1)!} + \cdots\right]$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots + \frac{z^{2j+1}}{(2j+1)!} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(2n+3)!}{(2n+1)!}\right| = \lim_{n\to\infty} (2n+3)(2n+2) = \infty$$

So the radius of convergence is ∞ thus the series converges for all $z \in \mathbb{C}$.