An Application of Fourier Transforms to Partial Differential Equations

Fourier transforms can be very useful in solving some partial differential equations.

Ex. Solve for the bounded solution, $\varphi(x,y)$, of Laplace's equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

for $-\infty<x<\infty,\ y>0$, with the boundary condition $\varphi(x,0)=h(x)$, where $\int_{-\infty}^{\infty}|h(x)|dx<\infty$ and $\int_{-\infty}^{\infty}|h(x)|^2dx<\infty$.

We start by taking the Fourier Transform of Laplace's equation with respect to x.

We know that: $\widehat{f^{(n)}}(k)=(ik)^n\hat{f}(k)$ so

$$\widehat{\frac{\partial^2\varphi}{\partial x^2}}(k,y)=-k^2\hat{\varphi}(k,y).$$

$$\widehat{\frac{\partial^2\varphi}{\partial y^2}}(k,y)=\int_{-\infty}^{\infty}\frac{\partial^2\varphi}{\partial y^2}(x,y)e^{-ikx}dx=\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty}\varphi(x,y)e^{-ikx}dx$$
$$=\frac{\partial^2\hat{\varphi}}{\partial y^2}(k,y).$$

Thus: $$\widehat{\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}}=-k^2\hat{\varphi}(k,y)+\frac{\partial^2\hat{\varphi}}{\partial y^2}(k,y)=0.$$

If k is constant, the above equation is just an ordinary differential equation in y. From elementary differential equations we know that the solutions of:

$$f''(t) - k^2 f(t) = 0; \qquad \text{where } k \text{ is a constant is}$$

$$f(t) = Ae^{kt} + Be^{-kt}.$$

So in our case the general solution to:

$$\frac{\partial^2 \hat{\varphi}}{\partial y^2}(k, y) - k^2 \hat{\varphi}(k, y) = 0 \qquad \text{is}$$

$$\hat{\varphi}(k, y) = A(k)e^{ky} + B(k)e^{-ky}.$$

We want to find a soution, $\varphi(x, y)$, to Laplace's equation that is bounded so $\hat{\varphi}(k, y)$ must be bounded. In order for

$$\hat{\varphi}(k, y) = A(k)e^{ky} + B(k)e^{-ky} \qquad \text{to be bounded we need}$$

$A(k) = 0$ for $k > 0$ and $B(k) = 0$ for $k < 0$.

Thus we can rewrite $\hat{\varphi}(k, y)$ as

$$\hat{\varphi}(k, y) = D(k)e^{-|k|y}.$$

Since the boundary condition is $\varphi(x, 0) = h(x)$ we have:

$$\hat{\varphi}(k, y) = \int_{-\infty}^{\infty} \varphi(x, y)e^{-ikx}dx\,; \qquad \text{so}$$

$$\hat{\varphi}(k, 0) = \int_{-\infty}^{\infty} \varphi(x, 0)e^{-ikx}dx = \int_{-\infty}^{\infty} h(x)e^{-ikx}dx = \hat{h}(k).$$

Plugging $y = 0$ into $\hat{\varphi}(k, y) = D(k)e^{-|k|y}$ we get

$$\hat{\varphi}(k, 0) = D(k) = \hat{h}(k).$$

So now we know that $\hat{\varphi}(k,y)$ satisfies:

$$\hat{\varphi}(k,y) = \hat{h}(k)e^{-|k|y}.$$

However, if $\quad f(x,y) = \frac{1}{\pi}(\frac{y}{x^2+y^2}),\;$ then $\;\hat{f}(k,y) = e^{-|k|y}.$

Thus $\quad \hat{\varphi}(k,y) = \hat{h}(k)\hat{f}(k,y).$

Now to find $\varphi(x,y)$ we use the Convolution Theorem:

if $\hat{\varphi} = \widehat{h * f} = \hat{h}\hat{f}$ then $\varphi(x,y) = h(x) * f(x,y).$

So we have:

$$\varphi(x,y) = h(x) * f(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{y}{(x-u)^2+y^2}\right)h(u)du.$$

If $h(x)$ is a Dirac delta function: $h(x) = \delta(x-w),$ then the solution becomes

$$\varphi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{y}{(x-u)^2+y^2}\right)\delta(u-w)du$$

$$\varphi(x,y) = \frac{1}{\pi}(\frac{y}{(x-u)^2+y^2}) \qquad \text{(since } \int_{-\infty}^{\infty} f(x)\delta(x-w)dx = f(w)\text{)}.$$

$G(x-w,y) = \frac{1}{\pi}(\frac{y}{(x-u)^2+y^2})$ is called a **Green's Function**.

Green's functions are solutions to differential equations with a delta function as the boundary value. If you have the Green's function for a differential equation you can construct a solution for a general boundary value, $h(x)$, by

$$\boldsymbol{\varphi(x,y) = \int_{-\infty}^{\infty} h(w)G(x-w,y)dw}.$$

Ex. Solve the time dependent (i.e. not steady state) heat flow problem given by

$$\frac{\partial \varphi}{\partial t}(x,t) = \frac{\partial^2 \varphi}{\partial x^2}(x,t),$$

with boundary condition $\varphi(x,0) = h(x)$.

First let's find the Green's function for this equation. Thus $\varphi(x,0) = \delta(x-w)$.

Take the Fourier transform of the partial differential equation with respect to x.

$$\widehat{\frac{\partial^2 \varphi}{\partial x^2}}(k,t) = -k^2 \hat{\varphi}(k,t)$$

$$\widehat{\frac{\partial \varphi}{\partial t}}(k,t) = \frac{\partial(\hat{\varphi}(k,t))}{\partial t}$$

So $$\frac{\partial(\hat{\varphi}(k,t))}{\partial t} = -k^2 \hat{\varphi}(k,t).$$

If k is constant then this is an ordinary differential equation in t.

Thus the solution is: $\hat{\varphi}(k,t) = \hat{\varphi}(k,0)e^{-k^2 t}$

(This is from first year Calculus: if $f'(t) = -k^2 f(t)$; then $f(t) = f(0)e^{-k^2 t}$).

$$\hat{\varphi}(k,0) = \int_{-\infty}^{\infty} \varphi(x,0)e^{-ikx}dx = \int_{-\infty}^{\infty} \delta(x-w)e^{-ikx}dx = e^{-ikw}$$

So $$\hat{\varphi}(k,t) = \hat{\varphi}(k,0)e^{-k^2 t} = e^{-ikw}e^{-k^2 t}.$$

Now take the inverse Fourier transform of $\hat{\varphi}(k,t)$, and that is $G(x-w,t)$.

$$G(x-w,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikw}e^{-k^2t}e^{ikx}\mathrm{dk}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-w)-k^2t}dk.$$

Now complete the square in the numerator.

$$-k^2t + i(x-w)k = -t(k^2 - \frac{i(x-w)}{t}k - \frac{(x-w)^2}{4t^2} + \frac{(x-w)^2}{4t^2})$$

$$= -t(k - \frac{i(x-w)}{2t})^2 - \frac{(x-w)^2}{4t}.$$

So $\quad G(x-w,t) = \frac{1}{2\pi}e^{-\frac{(x-w)^2}{4t}}\int_{-\infty}^{\infty} e^{-t\left(k-\frac{i(x-w)}{2t}\right)^2}dk.$

Now using the fact that $\int_{-\infty}^{\infty} e^{-tu^2}du = \frac{\sqrt{\pi}}{\sqrt{t}}$ we get

$$G(x-w,t) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{(x-w)^2}{4t}}.$$

Now that we have the Green's function $G(x-w,t)$, we can solve the partial differential equation for a given boundary function $\varphi(x,0) = h(x)$.

$$\varphi(x,t) = \int_{-\infty}^{\infty} h(w)G(x-w,t)dw$$

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} h(w)e^{-\frac{(x-w)^2}{4t}}dw.$$