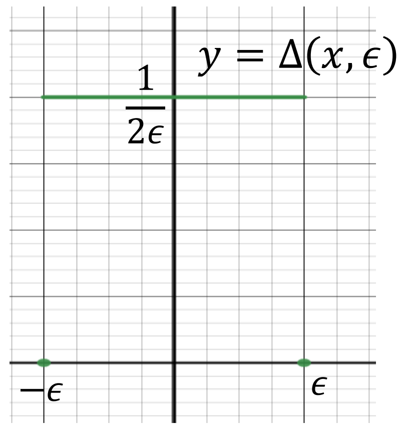
The Convolution Theorem

The Dirac Delta Function

The Dirac Delta function is a very unusual, but useful function.

Let $\Delta(x,\epsilon) = \frac{1}{2\epsilon}$ if $|x| < \epsilon$

$= 0$ if $|x| \geq \epsilon$.



If we take the Fourier transform of $\Delta(x,\epsilon)$ we get:

$$\hat{\Delta}(k,\epsilon) = \int_{-\infty}^{\infty} \Delta(x,\epsilon) e^{-ikx} dx$$

$$= \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} e^{-ikx} dx$$

$$= \left(\frac{1}{2\epsilon}\right) \frac{e^{-ikx}}{-ik}\Big|_{x=-\epsilon}^{x=\epsilon}$$

$$= \left(\frac{1}{2\epsilon}\right)\left(\frac{e^{-ik\epsilon} - e^{ik\epsilon}}{-ik}\right) = \left(\frac{1}{2\epsilon}\right)\left(\frac{-2i\sin(k\epsilon)}{-ik}\right) = \frac{\sin(k\epsilon)}{k\epsilon}.$$

Def. The **Dirac Delta Function** is defined as

$$\delta(x) = \lim_{\epsilon \to 0} \Delta(x, \epsilon).$$

So $\delta(x)$ is infinite at $x = 0$ and zero for $x \neq 0$.

$\delta(x - x_0)$ is infinite at $x = x_0$ and zero for $x \neq x_0$.

By $\int_{-\infty}^{\infty} \delta(x - x_0) dx = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \Delta(x - x_0, \epsilon)\, dx$, we will mean $\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \Delta(x - x_0, \epsilon) dx$. Hence we have:

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} \Delta(x - x_0, \epsilon) dx$$

$$= \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} \frac{1}{2\epsilon} dx = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} x \Big|_{x=x_0-\epsilon}^{x=x_0+\epsilon} = 1.$$

And if $f(x)$ is contiuous,

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \Delta(x - x_0, \epsilon)\, f(x) dx$$

$$= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \Delta(x - x_0, \epsilon) f(x) dx$$

$$= \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} \Delta(x - x_0, \epsilon) f(x) dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{x_0-\epsilon}^{x_0+\epsilon} f(x) dx = f(x_0). \quad (*)$$

The last equality comes from the fundamental theorem of calculus (FTC). If

$$F(x) = \int_0^x f(t) dt$$

then the LHS of $(*)$ above is just $F'(x_0)$ and by the FTC equals $f(x_0)$.

In particular, if $f(x) = e^{-ikx}$, then $f(0) = 1$ and we get:

$$\hat{\delta}(k) = \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx$$
$$= e^{-ik(0)} = 1.$$

To solve differential equations and partial differential equations using Fourier transforms we need the following very useful relationship between the Fourier transform of $f'(x)$ and the Fourier transform of $f(x)$.

$$\widehat{f'}(k) = \int_{-\infty}^{\infty} f'(x) e^{-ikx} dx$$

Now integrate by parts: $u = e^{-ikx}$ $\qquad v = f(x)$

$$du = -ike^{-ikx} \qquad dv = f'(x)dx$$

$$\widehat{f'}(k) = e^{-ikx} f(x)\Big|_{x=-\infty}^{x=\infty} + ik \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

Since we have assumed that $\lim_{x \to \pm\infty} f(x) = 0$, we have:

$$\widehat{f'}(k) = ik \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

i.e. $\boldsymbol{\widehat{f'}(k) = (ik)\hat{f}(k)}$.

Repeating this argument gives us:

$$\boldsymbol{\widehat{f^{(n)}}(k) = (ik)^n \hat{f}(k)}.$$

Another important relationship of Fourier transforms comes from the following:

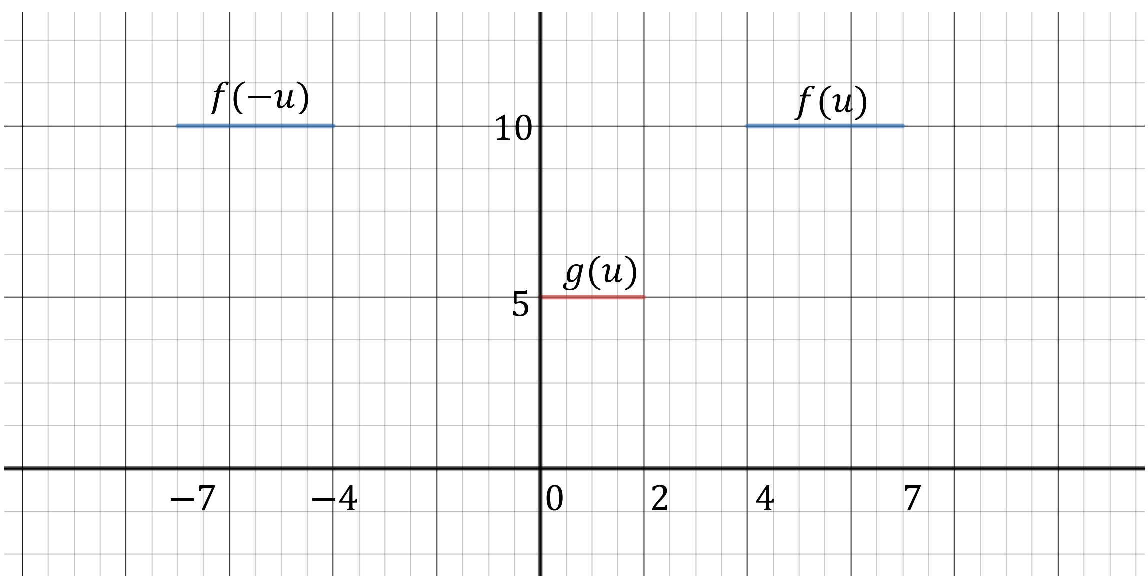
Def. The **convolution** of $f(x)$ and $g(x)$ is defined to be:

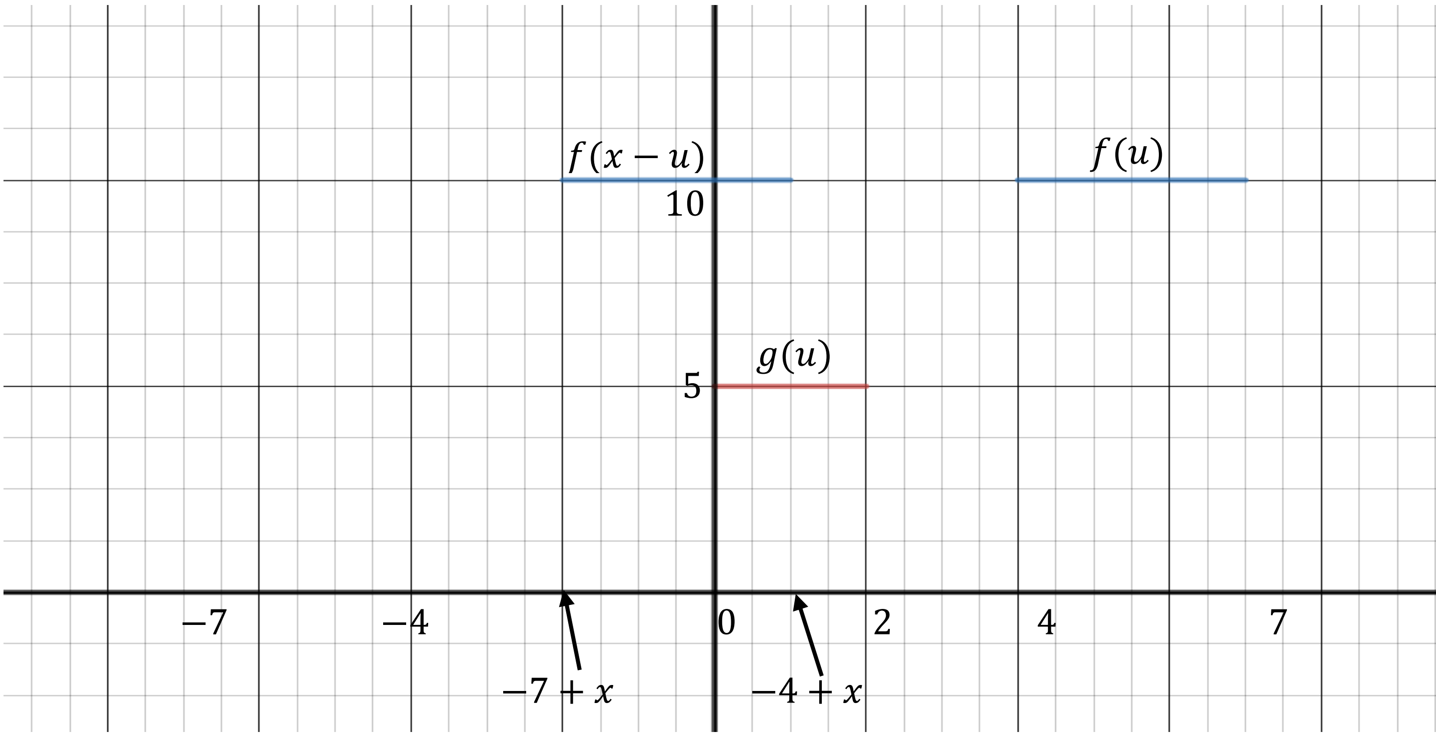
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-u)g(u)du = \int_{-\infty}^{\infty} f(u)g(x-u)du$$
$$= (g * f)(x)$$

To get the middle equality let $w = x - u$ and $dw = -du$.

Notice that if we have the graph of a function $y = f(u)$, then the graph of $y = f(-u)$ the reflection of this graph about the y axis. Thus the graph of $y = f(x-u)$ is the translation of the graph of $y = f(-u)$ by x units to the right.

Ex. Let $f(x) = 10,$ if $4 \leq x \leq 7$ and zero otherwise and $g(x) = 5,$ if $0 \leq x \leq 2$ and zero otherwise. Find $(f * g)(x)$.



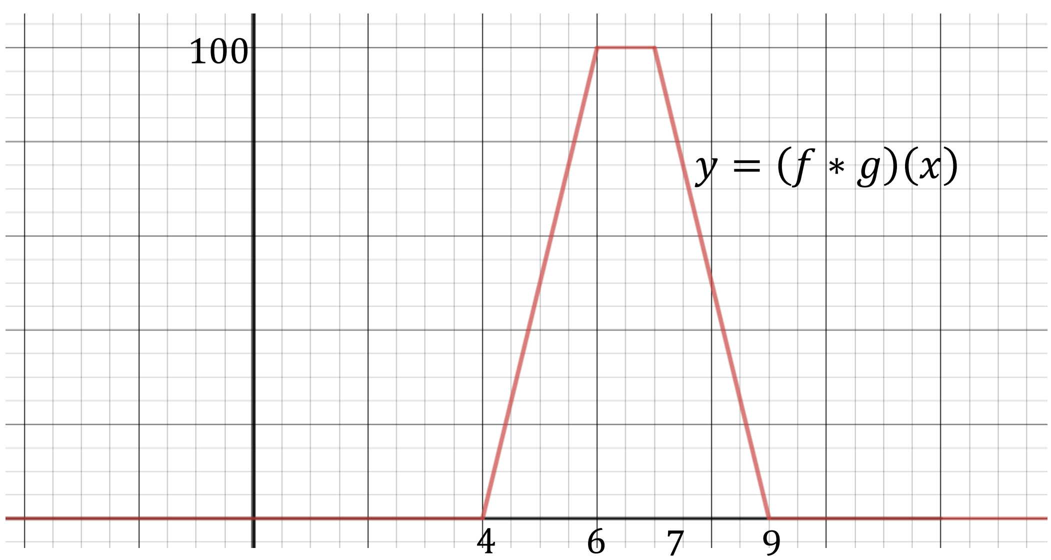


Notice that if $x < 4$ or $x > 9$, $f(x-u)g(u) = 0$. When $4 \le x \le 9$ we have:

$$(f * g)(x) = \int_{u=0}^{u=-4+x} 50du = 50x - 200, \qquad \text{if } 4 \le x < 6$$

$$(f * g)(x) = \int_{u=0}^{u=2} 50du = 100, \qquad \text{if } 6 \le x < 7$$

$$(f * g)(x) = \int_{u=-7+x}^{u=2} 50du = 450 - 50x, \qquad \text{if } 7 \le x \le 9.$$



The Convolution Theorem: $\widehat{f*g}(k) = \hat{f}(k)\hat{g}(k)$.

Proof:

$$\widehat{f*g}(k) = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x-u)g(u)du)e^{-ikx}dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u)g(u)e^{-ikx}dudx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u)g(u)e^{-iku}e^{-ik(x-u)}dudx$$

Now let $w = x - u$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(w)g(u)e^{-iku}e^{-ik(w)}dudw$$

$$= \int_{-\infty}^{\infty} g(u)e^{-iku}du \int_{-\infty}^{\infty} f(u)\, e^{-ik(w)}dw$$

$$= \hat{f}(k)\hat{g}(k).$$

Thus we have the following relationship between the inverse Fourier transform of $\hat{f}(k)\hat{g}(k)$ and $(f*g)(x)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)e^{ikx}dk = (f*g)(x).$$

Ex. Let $f(x) = 1$ if $-1 \leq x \leq 1$ and 0 otherwise. Show the convolution theorem is satisfied for $g(x) = f(x)$ by calculating each side. That is, show that $\widehat{f * f}(k) = \hat{f}(k)\left(\hat{f}(k)\right)$.

Let's start by calculating $\hat{f}(k)$.

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \int_{-1}^{1} e^{-ikx}\, dx$$

$$= \left(\frac{1}{-ik}\right) e^{-ikx}\Big|_{x=-1}^{x=1} = \frac{i}{k}\left(e^{ik} - e^{-ik}\right) = \frac{2}{k}\sin(k).$$

$$(f * f)(x) = \int_{-\infty}^{\infty} f(u) f(x-u) du$$

$$(f * f)(x) = \int_{-1}^{1} f(x-u) du = \int_{u=-1}^{u=x+1} 1 du = x + 2 \qquad \text{if } -2 \leq x < 0$$

$$(f * f)(x) = \int_{-1}^{1} f(x-u) du = \int_{u=-1+x}^{u=1} 1 du = 2 - x \qquad \text{if } 0 \leq x \leq 2$$

$$(f * f)(x) = 0 \qquad \text{if } |x| > 2.$$

Thus we have:

$$\widehat{f * f}(k) = \int_{-\infty}^{\infty} (f * f)(x) e^{-ikx} dx$$

$$= \int_{-2}^{0} (x+2) e^{-ikx} dx + \int_{0}^{2} (2-x) e^{-ikx} dx.$$

In the second integral make the substitution $v = -x, \quad dv = -dx.$

$$= \int_{-2}^{0} (x+2) e^{-ikx} dx - \int_{0}^{-2} (2+v) e^{ikv} dv$$

$$= \int_{-2}^{0}(x+2)e^{-ikx}dx + \int_{-2}^{0}(2+v)e^{ikv}dv.$$

Now replace v with x in the second integral to get:

$$= \int_{-2}^{0}(x+2)(e^{-ikx}+e^{ikx})dx$$

$$= \int_{-2}^{0}(x+2)(2coskx)dx.$$

Now integrate by parts with $u = (x+2)$ and $dv = 2\cos(kx)\,dx$.

$$= \frac{2}{k^2}(1-cos2k) = \frac{4}{k^2}\sin^2(k) = \hat{f}(k)\left(\hat{f}(k)\right).$$

Ex. Solve the following differential equations using Fourier transforms:

$$\frac{d^4u}{dx^4} - 2w^2\frac{d^2u}{dx^2} + w^4u = f(x); \qquad w > 0 \text{ and}$$

$$\lim_{x\to\pm\infty} u(x) = 0.$$

Recall that when we take the Fourier transform (FT) of a derivative of a function we get: $\widehat{f^{(n)}}(k) = (ik)^n\hat{f}(k).$

So we have: $\widehat{\frac{d^4u}{dx^4}} = (ik)^4\widehat{U}(k) = k^4\widehat{U}(k);$ where $\widehat{U}(k)$ is the FT of $u(x)$.

and $\widehat{\frac{d^2u}{dx^2}} = (ik)^2\widehat{U}(k) = -k^2\widehat{U}(k).$

So when we take the FT of both sides of the differential equation we get:

$$k^4\widehat{U}(k) + 2w^2k^2\widehat{U}(k) + w^4\widehat{U}(k) = \widehat{F}(k)$$

where $\widehat{F}(k)$ is the FT of $f(x)$.

Factoring out the $\widehat{U}(k)$ we get:

$$(k^4 + 2w^2k^2 + w^4)\widehat{U}(k) = \widehat{F}(k)$$

$$(k^2 + w^2)^2\widehat{U}(k) = \widehat{F}(k)$$

$$\widehat{U}(k) = \frac{\widehat{F}(k)}{\left(k^2+w^2\right)^2} \; .$$

With a straight forward calculation it can be shown that if

$$\widehat{G}(k) = \frac{1}{\left(k^2+w^2\right)^2}$$

then $g(x)$, the inverse FT of $\widehat{G}(k)$ is given by

$$g(x) = \frac{1}{4w^3}(1 + w|x|)e^{-w|x|}.$$

So if we think of $\widehat{U}(k)$=$\widehat{F}(k)\widehat{G}(k)$

we know that $u(x) = (f * g)(x)$

$$u(x) = \frac{1}{4w^3}\int_{-\infty}^{\infty}(1 + w|x - u|)e^{-w|x-u|}f(u)du$$

where $u(x)$ is a solution to the original differential equation.

Notice that if $f(x) = \delta(x)$ in the last example we would get:

$$\widehat{U}(k)=\hat{\delta}(k)\widehat{G}(k) = \widehat{G}(k), \text{ since } \hat{\delta}(k) = 1.$$

Thus we have: $u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(k^2+w^2)^2}e^{ikx}dk$

$$= \frac{1}{4w^3}(1 + w|x|)e^{-w|x|} = g(x).$$