The Convolution Theorem- HW Problems

1. Let $f(x) = 1$ if $-2 \le x \le 2$ and 0 otherwise. Show that

a. $\hat{f}(k) = \frac{2}{k}\sin(2k)$

b. $(f * f)(x) = x + 4$ if $-4 \le x \le 0$

                     $= 4 - x$ if $0 < x \le 4$

                     $= 0$ if $|x| > 4$.

c. $\widehat{f * f}(k) = \frac{4}{k^2}\sin^2(2k)$ and thus $\widehat{f * f}(k) = (\hat{f}(k))(\hat{f}(k))$ and hence satisfies the convolution theorem.

Hint: $\sin^2(2k) = \frac{1}{2} - \frac{1}{2}\cos(4k)$.

2. Solve the following differential equation by taking the Fourier transform of the equation and then using the convolution theorem.

$$\frac{d^2u}{dx^2} - u = -f(x); \quad (\text{assume } \lim_{x \to \pm\infty} u(x) = 0).$$

You can write the solution in terms of an integral which you don't need to evaluate.