Fourier Transforms

The Fourier transform of a real valued function $f(x)$ is another function called $\hat{F}(k)$ (where k is a real variable) given by:

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

The inverse Fourier transform of a function $\hat{F}(k)$ is given by:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)e^{ikx}dk$$

It turns out that at points where $f(x)$ is continuous the above equation holds (i.e. the inverse Fourier transform of the Fourier transform of $f(x)$ equals $f(x)$). At points where $f(x)$ is discontinuous, say x_0, we have:

$$\lim_{\epsilon\to 0}\frac{1}{2}(f(x_0+\epsilon)+f(x_0-\epsilon)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)e^{ikx_0}dk.$$

That is, the RHS converges to the average of the limit of $f(x)$ from the right and from the left. We will assume that $f(x)$ has at most a finite number of discontinuities, $\lim_{x\to\pm\infty} f(x) = 0$ and $\int_{-\infty}^{\infty} |f(x)|dx$, $\int_{-\infty}^{\infty}|f(x)|^2dx$ exist.

The Fourier transform (on a finite interval) at integer values of x shows up as the coefficients of a function's Fourier series. That is, if you have a function $f(x)$ defined on $(-L, L)$, we can extend it as a periodic function of period $2L$ and the Fourier series (which we will not be studying here) of that function is:

$$f(x) = \sum_{n=-\infty}^{n=\infty} \hat{F}(n)e^{(\frac{n\pi xi}{L})}$$

where $\hat{F}(n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-(\frac{in\pi x}{L})}dx$.

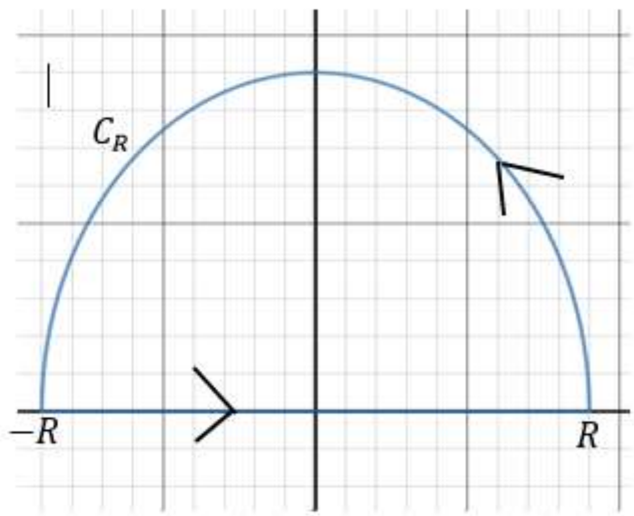
Ex. Compute the Fourier transform, $\hat{F}(k)$, of $f(x) = \frac{1}{x^2+4}$ and show that

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)e^{ikx}dk$ where $\hat{F}(k) = \int_{-\infty}^{\infty} \frac{1}{x^2+4}e^{-ikx}dx.$

First let's calculate $\hat{F}(k) = \int_{-\infty}^{\infty} \frac{1}{x^2+4}e^{-ikx}dx$. We will do this through contour integration when $k \neq 0$, however, we will need to consider separately the cases where $k < 0$ and $k > 0$.

Case 1: $k < 0$. Then $-k > 0$. This determines which semicircular region we integrate around (upper half plane or lower half plane). In this case we use the semicircular region in the upper half plane .

$C = C_R + [-R, R]$



$$\lim_{R\to\infty} \oint_C \frac{1}{z^2+4}e^{-ikz}dz = \lim_{R\to\infty}[\int_{C_R} \frac{1}{z^2+4}e^{-ikz}dz + \int_{-R}^{R} \frac{1}{x^2+4}e^{-ikx}dx].$$

$\lim_{R\to\infty} \int_{C_R} \frac{1}{z^2+4}e^{-ikz}dz = 0$ by Jordan's lemma (this is why we chose the upper half plane), since $\frac{1}{z^2+4}$ is a rational function where the degree of the denominator is larger than the degree of the numerator, it goes to 0 uniformly as R goes to infinity.

So now we have:

$$\lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+4} e^{-ikx} dx = \int_{-\infty}^{\infty} \frac{1}{x^2+4} e^{-ikx} dx.$$

We can evaluate the integral on the LHS by Cauchy's residue theorem.

$$\lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = 2\pi i(sum\ of\ residues\ of\ poles\ inside\ C)$$

$\frac{1}{z^2+4} e^{-ikz}$ has poles at $z = \pm 2i$, but only $z = 2i$ is inside C.

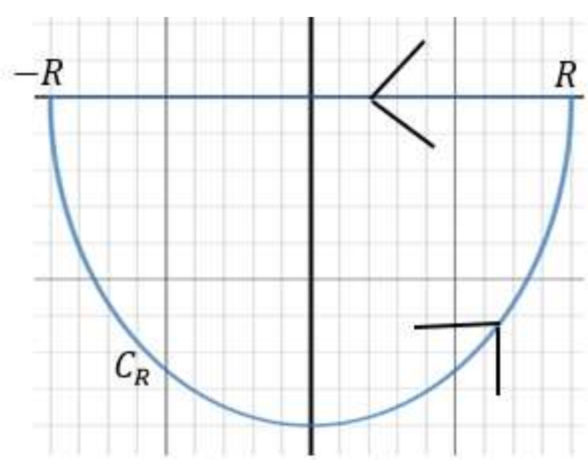
$$Res\left(\frac{1}{z^2+4} e^{-ikz}; 2i\right) = \lim_{z\to 2i} (z-2i)\left(\frac{e^{-ikz}}{(z-2i)(z+2i)}\right) = \frac{e^{-ik(2i)}}{(2i+2i)} = \frac{e^{2k}}{4i}$$

So

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4} e^{-ikx} dx = \lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = 2\pi i(\frac{e^{2k}}{4i}) = \frac{\pi}{2} e^{2k}$$

Thus $\quad \hat{F}(k) = \frac{\pi}{2} e^{2k}$, if $k < 0$.

Case 2: If $k > 0$, then $-k < 0$, so we need to use the semicircular region in the lower half plane in order to use Jordan's lemma.



$$\lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = \lim_{R\to\infty} [\int_{C_R} \frac{1}{z^2+4} e^{-ikz} dz + \int_R^{-R} \frac{1}{x^2+4} e^{-ikx} dx].$$

Now by Jordan's lemma (as used in the first part)

$$\lim_{R\to\infty} \int_{C_R} \frac{1}{z^2+4} e^{-ikz} dz = 0$$

and $\lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = \lim_{R\to\infty} \int_R^{-R} \frac{1}{x^2+4} e^{-ikx} dx = -\int_{-\infty}^{\infty} \frac{1}{x^2+4} e^{-ikx} dx.$

$$\lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = 2\pi i(sum\ of\ residues\ of\ poles\ inside\ C).$$

$\frac{1}{z^2+4} e^{-ikz}$ has poles at $z = \pm 2i$, but only $z = -2i$ is inside C.

$$Res\left(\frac{1}{z^2+4} e^{-ikz}; -2i\right) = \lim_{z\to -2i} (z+2i)\left(\frac{e^{-ikz}}{(z-2i)(z+2i)}\right) = \frac{e^{-ik(-2i)}}{(-2i-2i)} = \frac{e^{-2k}}{-4i}.$$

So we have:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4} e^{-ikx} dx = -\lim_{R\to\infty} \oint_C \frac{1}{z^2+4} e^{-ikz} dz = -2\pi i(\frac{e^{-2k}}{-4i}) = \frac{\pi}{2} e^{-2k}$$

Thus $\hat{F}(k) = \frac{\pi}{2} e^{-2k}$, if $k > 0$.

Putting this together with the result $\hat{F}(k) = \frac{\pi}{2} e^{2k}$, if $k < 0$, we get:

$\hat{F}(k) = \frac{\pi}{2} e^{-2|k|}$; for $k \neq 0$.

If $k = 0$, $\int_{-\infty}^{\infty} \frac{1}{x^2+4} dx$ can be computed directly by:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4} dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{x^2+4} dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{x^2+4} dx$$

$$= \frac{1}{4}[\lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+(\frac{x}{2})^2} dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+(\frac{x}{2})^2} dx]; \text{ now let } u = \frac{x}{2}$$

$$= \frac{1}{2}[\lim_{a\to-\infty} \int_{\frac{a}{2}}^{0} \frac{1}{1+(u)^2} du + \lim_{b\to\infty} \int_{0}^{\frac{b}{2}} \frac{1}{1+(u)^2} du] = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{2}$$

.

So $\hat{F}(k) = \frac{\pi}{2} e^{-2|k|}; \quad k \in \mathbb{R}$.

Now let's show: $\frac{1}{x^2+4} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (\frac{\pi}{2} e^{-2|k|}) dk$.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} (\frac{\pi}{2} e^{-2|k|}) dk = \frac{1}{2\pi} \int_{-\infty}^{0} e^{ikx} (\frac{\pi}{2} e^{2k}) dk + \frac{1}{2\pi} \int_{0}^{\infty} e^{ikx} (\frac{\pi}{2} e^{-2k}) dk$$

$$= \frac{1}{4} [\int_{-\infty}^{0} e^{(2+ix)k} dk + \int_{0}^{\infty} e^{(-2+ix)k} dk]$$

$$= \frac{1}{4} [\frac{e^{(2+ix)k}}{2+ix}\Big|_{k=-\infty}^{k=0} + \frac{e^{(-2+ix)k}}{-2+ix}\Big|_{k=0}^{k=\infty}$$

$$= \frac{1}{4} [\left(\frac{1}{2+ix} - 0\right) + \left(0 - \frac{1}{-2+ix}\right)]$$

$$= \frac{1}{4}\left(\frac{1}{2+ix} - \frac{1}{-2+ix}\right) = \frac{1}{4} (\frac{-4}{-4-x^2}) = \frac{1}{4+x^2} .$$

Ex. Find the Fourier transform, $\hat{F}(k)$, of : $f(x) = 1 \quad 0 \le x \le 1$

$$= 0 \quad x < 0 \ \ or \ \ x > 1$$

And then find the inverse Fourier transform of $\hat{F}(k)$.

$$\hat{F}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{0}^{1} e^{-ikx}dx = \frac{e^{-ikx}}{-ik}\Bigg|_{x=0}^{x=1}$$

$$\hat{F}(k) = \frac{i}{k}\left(e^{-ik} - 1\right).$$

Now let's evaluate the inverse Fourier transform of $\hat{F}(k)$.

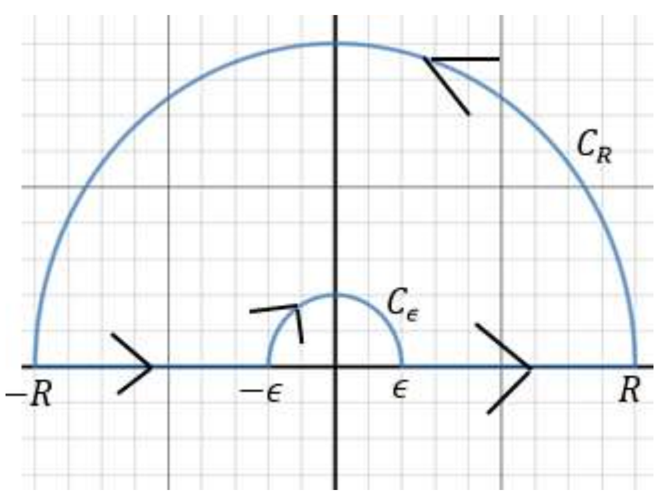
$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{k}(e^{-ik} - 1)e^{ikx}dk$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{1}{k}\left(e^{i(x-1)k} - e^{ikx}\right)dk.$$

Let's break this up into 2 integrals.

Let's evaluate $\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(x-1)k}}{k}dk$ first.

Case 1: $x - 1 > 0$. The integrand has a pole at $k = 0$, so let's use a contour integral with 2 semicircles:



$$C = C_R + [-R, -\epsilon] + C_\epsilon + [\epsilon, R]$$

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{i(x-1)z}}{z} dz$$

$$= \lim_{R\to\infty,\epsilon\to 0} [\int_{C_R} \frac{e^{i(x-1)z}}{z} dz + \int_{-R}^{\epsilon} \frac{e^{i(x-1)k}}{k} dk + \int_{C_\epsilon} \frac{e^{i(x-1)z}}{z} dz + \int_{\epsilon}^{R} \frac{e^{i(x-1)k}}{k} dk].$$

$\lim_{R\to\infty} \oint_{C_R} \frac{e^{i(x-1)z}}{z} dz = 0$ by Jordan's lemma because $x - 1 > 0$, and $\frac{1}{z}$ is a rational function whose denominator has a higher degree than its numerator so it converges uniformly to 0 as R goes to infinity.

To evaluate $\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{i(x-1)z}}{z} dz$;

let $z = \epsilon e^{i\theta}, \quad dz = i\epsilon e^{i\theta} d\theta$

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{i(x-1)z}}{z} dz = \lim_{\epsilon\to 0} \int_{\pi}^{0} \frac{e^{i(x-1)\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta$$

$$= \lim_{\epsilon\to 0} i \int_{\pi}^{0} e^{i(x-1)\epsilon e^{i\theta}} d\theta = i \int_{\pi}^{0} \lim_{\epsilon\to 0} e^{i(x-1)\epsilon e^{i\theta}} d\theta$$

$$= i \int_{\pi}^{0} 1 d\theta = -\pi i.$$

Note: we can pass the $\lim_{\epsilon\to 0}$ through the integral sign because $e^{i(x-1)\epsilon e^{i\theta}}$ converges to 1 uniformly as ϵ goes to 0.

So we have:

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{i(x-1)z}}{z} dz = -\pi i + \int_{-\infty}^{\infty} \frac{e^{i(x-1)k}}{k} dk.$$

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{i(x-1)z}}{z} dz = 2\pi i(sum\ of\ residues\ of\ poles\ inside\ C)$$

But there are no poles inside C so

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{i(x-1)z}}{z} dz = 0\ .$$

Thus: $\quad 0 = -\pi i + \int_{-\infty}^{\infty} \frac{e^{i(x-1)k}}{k} dk$ and

$\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(x-1)k}}{k} dk = \left(\frac{i}{2\pi}\right)\pi i = -\frac{1}{2}$ for $x - 1 > 0 \quad or\ x > 1$.

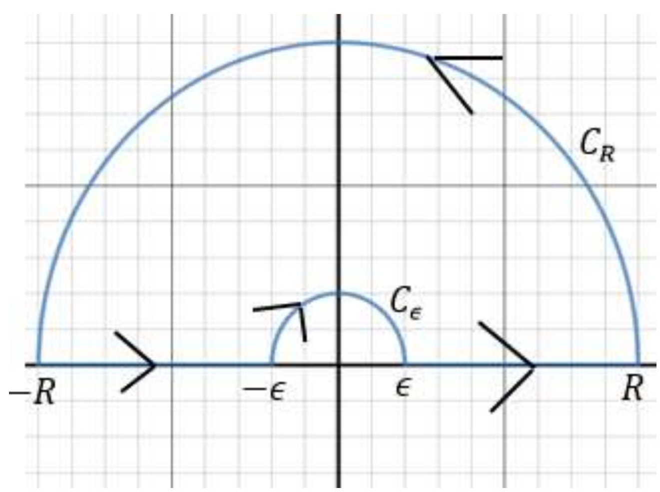
Case 2: If $x - 1 < 0$; i.e. $x < 1$, a very similar argument using the analogous contour but in the lower half plane we get:

$\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(x-1)k}}{k} dk = \frac{1}{2}$ for $x < 1$.

Now let's evaluate the second integral: $\frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k} dk.$

Again we need to consider 2 cases: when $x > 0$ and when $x < 0$.

Case 1: $x > 0$; we use the contour with 2 semicircles in the upper half plane.



Once again we have:

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{ixz}}{z} dz$$

$$= \lim_{R\to\infty,\epsilon\to 0} [\int_{C_R} \frac{e^{ixz}}{z} dz + \int_{-R}^{\epsilon} \frac{e^{ixk}}{k} dk + \int_{C_\epsilon} \frac{e^{ixz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{ixk}}{k} dk].$$

$\lim_{R\to\infty} \int_{C_R} \frac{e^{ixz}}{z} dz = 0$ by Jordan's lemma because $x > 0$, and $\frac{1}{z}$ is a rational function whose denominator has a higher degree than its numerator so it converges uniformly to 0 as R goes to infinity.

To evaluate $\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{ixz}}{z} dz$;

let $z = \epsilon e^{i\theta}, \quad dz = i\epsilon e^{i\theta} d\theta$

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{ixz}}{z} dz = \lim_{\epsilon\to 0} \int_{\pi}^{0} \frac{e^{ix\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta$$

$$= \lim_{\epsilon\to 0} i \int_{\pi}^{0} e^{ix\epsilon e^{i\theta}} d\theta = i \int_{\pi}^{0} \lim_{\epsilon\to 0} e^{ix\epsilon e^{i\theta}} d\theta$$

$$= i \int_{\pi}^{0} 1 d\theta = -\pi i.$$

So we have:

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{ixz}}{z} dz = -\pi i + \int_{-\infty}^{\infty} \frac{e^{ixk}}{k} dk.$$

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{ixz}}{z} dz = 2\pi i(sum\ of\ residues\ of\ poles\ inside\ C).$$

But there are no poles inside C so

$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{ixz}}{z} dz = 0.$$

Thus: $0 = -\pi i + \int_{-\infty}^{\infty} \frac{e^{ixk}}{k} dk$ and

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixk}}{k} dk = \left(\frac{i}{2\pi}\right) \pi i = -\frac{1}{2} \quad \text{for } x > 0\,.$$

Case 2: If $x < 0$, we use the contour with two semicircles, but in the lower half plane (to make Jordan’s lemma work).

A very similar argument to the one just made gives us

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixk}}{k} dk = \left(\frac{i}{2\pi}\right)(-\pi i) = \frac{1}{2} \quad \text{for } x < 0\,.$$

Thus we have:

$$g(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k} (e^{i(x-1)k} - e^{ikx}) dk$$

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(x-1)k}}{k} dk = -\frac{1}{2} \quad \text{if } x > 1$$

$$= \frac{1}{2} \quad \text{if } x < 1$$

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixk}}{k} dk = -\frac{1}{2} \qquad \text{if } \quad x > 0$$

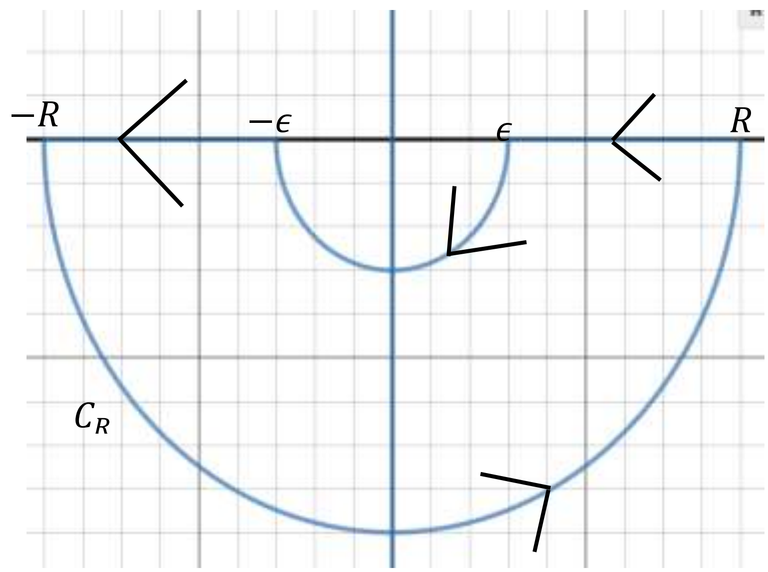
$$= \frac{1}{2} \qquad \text{if } \quad x < 0.$$

So $$g(x) = 1 \quad \text{if } 0 < x < 1$$
$$= 0 \quad \text{if } x < 0 \text{ or } x > 1.$$

But what are $g(0)$ and $g(1)$?

$$g(0) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ik}-1}{k}\,dk.$$

If we use a contour with 2 semicircles in the lower half plane get



$$\lim_{R\to\infty,\epsilon\to 0} \oint_C \frac{e^{-iz}-1}{z}\,dz$$

$$= \lim_{R\to\infty,\epsilon\to 0}\left[\int_{C_R} \frac{e^{-iz}-1}{z}\,dz + \int_R^{\epsilon} \frac{e^{-iz}-1}{z}\,dk + \int_{C_\epsilon} \frac{e^{-iz}-1}{z}\,dz + \int_{-\epsilon}^{-R} \frac{e^{-iz}-1}{z}\,dk\right]$$

The LHS is 0 because there are no poles inside of C.

$$\lim_{R\to\infty,\epsilon\to 0} \int_{C_R} \frac{e^{-iz}-1}{z}\,dz = \lim_{R\to\infty,\epsilon\to 0}\left[\int_{C_R} \frac{e^{-iz}}{z}\,dz - \int_{C_R} \frac{1}{z}\,dz\right]$$

$\lim_{R\to\infty,\epsilon\to 0} \int_{C_R} \frac{e^{-iz}}{z} dz = 0$ by Jordan's lemma.

$$\lim_{R\to\infty,\epsilon\to 0} \int_{C_R} \frac{1}{z} dz = \lim_{R\to\infty,\epsilon\to 0} \int_{\pi}^{2\pi} \frac{1}{Re^{i\theta}} \left(iRe^{i\theta}\right) d\theta = \int_{\pi}^{2\pi} i d\theta = \pi i$$

So $\lim_{R\to\infty,\epsilon\to 0} \int_{C_R} \frac{e^{-iz}-1}{z} dz = 0 - \pi i = -\pi i.$

$$\lim_{R\to\infty,\epsilon\to 0} \int_{C_\epsilon} \frac{e^{-iz}-1}{z} dz = \lim_{R\to\infty,\epsilon\to 0} \int_{0}^{-\pi} \frac{e^{-i\epsilon e^{i\theta}}-1}{\epsilon e^{i\theta}} \left(i\epsilon e^{i\theta}\right) d\theta$$

$$= \lim_{R\to\infty,\epsilon\to 0} i \int_{0}^{-\pi} (e^{-i\epsilon e^{i\theta}} - 1) d\theta$$

$$= i \int_{0}^{-\pi} \lim_{R\to\infty,\epsilon\to 0} (e^{-i\epsilon e^{i\theta}} - 1) d\theta$$

$$= 0.$$

So $0 = -\pi i + 0 + \int_{\infty}^{-\infty} \frac{e^{-ik}-1}{k} dk$ or $\int_{-\infty}^{\infty} \frac{e^{-ik}-1}{k} dk = -\pi i.$

Thus $g(0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik}-1}{k} dk = \left(\frac{i}{2\pi}\right)(-\pi i) = \frac{1}{2}$.

For $g(1) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{ik}}{k} dk$; we use the contour with 2 semicircles in the upper half plane.

A very similar argument to the one used to get $g(0) = \frac{1}{2}$, give us $g(1) = \frac{1}{2}$.

So we have the following description of $g(x)$;

$$g(x) = 1 \qquad \text{if } 0 < x < 1$$

$$= 0 \qquad \text{if } x < 0 \text{ or } x > 1$$

$$g(0) = \frac{1}{2} = \frac{1}{2}(\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x))$$

$$g(1) = \frac{1}{2} = \frac{1}{2}(\lim_{x\to 1^+} f(x) + \lim_{x\to 1^-} f(x)) .$$

So $g(x)$, which is what we get when we take the inverse Fourier transform of the the Fourier transform of $f(x)$, is equal to $f(x)$ whereever $f(x)$ is continuous, and equal to the average or the limit from the right and left of $f(x)$ at points of discontinuity.