The Cauchy Residue Theorem

Let $f(z)$ be analytic in a region D defined by $0 < |z - z_0| < R$ and $z = z_0$ is an isolated singularity of $f(z)$. We know that $f(z)$ has a Laurent series given by:

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n (z - z_0)^n \,; \qquad \text{with } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^n} dz$$

where C is a simple closed contour lying in D and enclosing z_0.

$\sum_{n=-\infty}^{n=-1} a_n (z - z_0)^n$ is called the principal part of the series and a_{-1}, the coefficient of the $\frac{1}{z - z_0}$ term is called the residue of $f(z)$ at $z = z_0$.

We will denote the residue of $f(z)$ at $z = z_0$ by $Res(f(z);\ z_0)$.

We also know that for a simple closed curve C where z_0 is inside of C:

$$\oint_C \frac{1}{(z - z_0)^n} dz = 0 \qquad if\ n \neq 1$$

$$= 2\pi i \qquad if\ n = 1\,.$$

Thus we saw that: $\oint_C f(z) dz = \oint_C \sum_{n=-\infty}^{n=\infty} a_n (z - z_0)^n \, dz = 2\pi i (\, a_{-1})$.

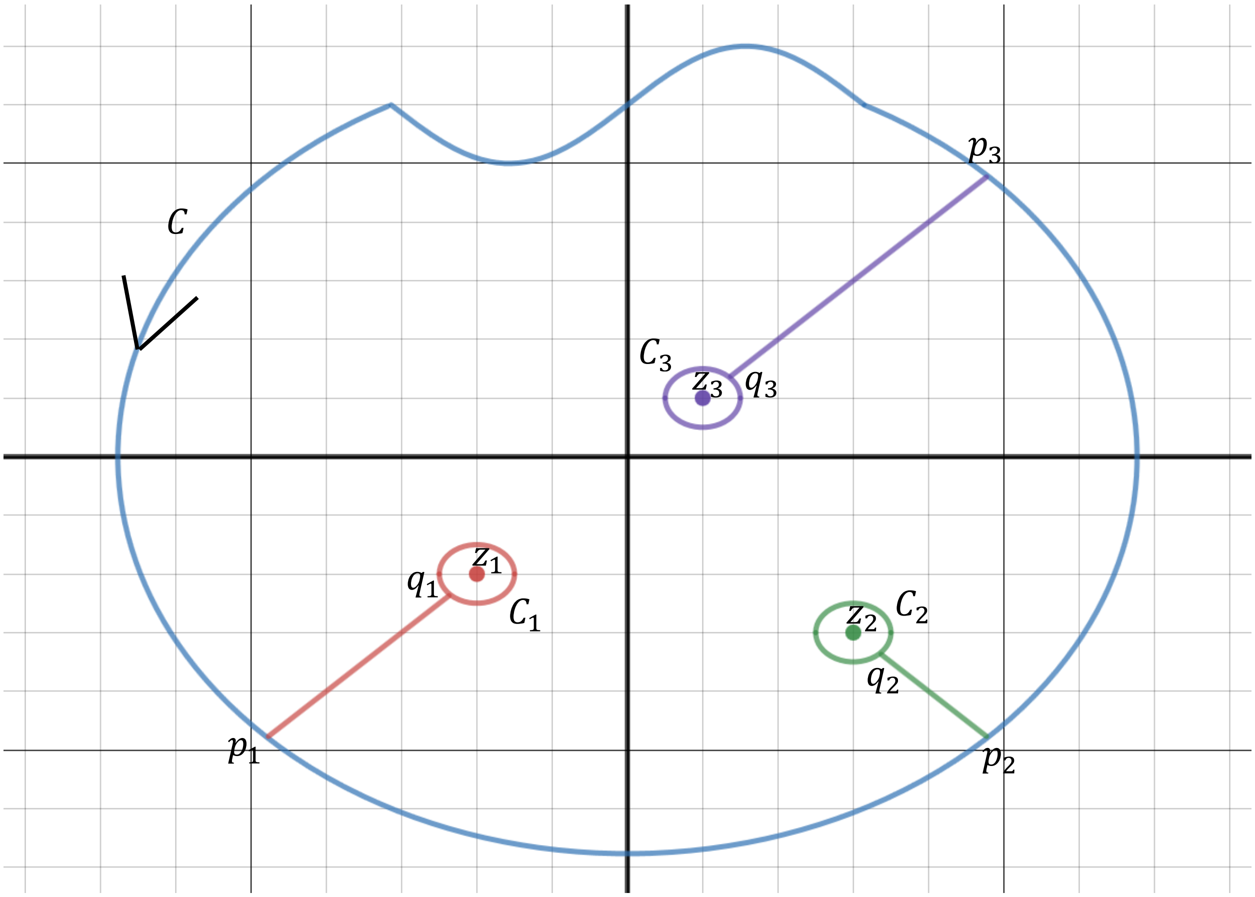
If there are multiple singularities in D, $z_1, z_2, \ldots, z_n$, inside of C we can extend this result as follows:

Theorem (Cauchy Residue Theorem); Let $f(z)$ be analytic inside a simple closed contour C, except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ located inside C. Then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^{n} c_j$$

where c_j is the residue of $f(z)$ at $z = z_j$.

Proof: Let $C_1, C_2, \ldots, C_n$ be small nonintersecting circles centered at $z_1, z_2, \ldots, z_n$. Create crosscuts from C to $C_1, C_2, \ldots, C_n$ (see diagram below).



Since $\int_{p_j}^{q_j} f(z)dz + \int_{q_j}^{p_j} f(z)dz = 0$, for each j we have:

$$\oint_{\Gamma} f(z)dz = 0; \quad \text{where } \Gamma = C - C_1 - C_2 \ldots - C_n;$$

by Cauchy's theorem.

Thus $\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz.$

Using the Laurent expansion for $f(z)$ around each singularity we get:

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} c_j$$

where c_j is the residue of $f(z)$ at $z = z_j$.

Ex. Evaluate $\frac{1}{2\pi i}\oint_C z^2 e^{\frac{1}{z}}dz$; where C is the unit circle $|z| = 1$.

The only singularity inside of C is the point $z = 0$. So we need the Laurent series around the point $z = 0$.

$$z^2 e^{\frac{1}{z}} = z^2\left(1 + \frac{1}{z} + \frac{1}{(2!)(z^2)} + \frac{1}{(3!)(z^3)} + \cdots + \frac{1}{(n!)(z^n)} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \cdots + \frac{1}{(n!)(z^{n-2})} + \cdots$$

Hence the $Res(f(z);0) = \frac{1}{6}$.

Thus $\frac{1}{2\pi i}\oint_C z^2 e^{\frac{1}{z}}dz = \frac{1}{6}$.

Ex. Evaluate $\oint_C \frac{2z+6}{z^2+2z}\,dz$ where C is the circle

a. $|z| = 1$
b. $|z+2| = 1$
c. $|z| = 3$.

First use partial fractions to get the Laurent series for $\frac{2z+6}{z^2+2z}$.

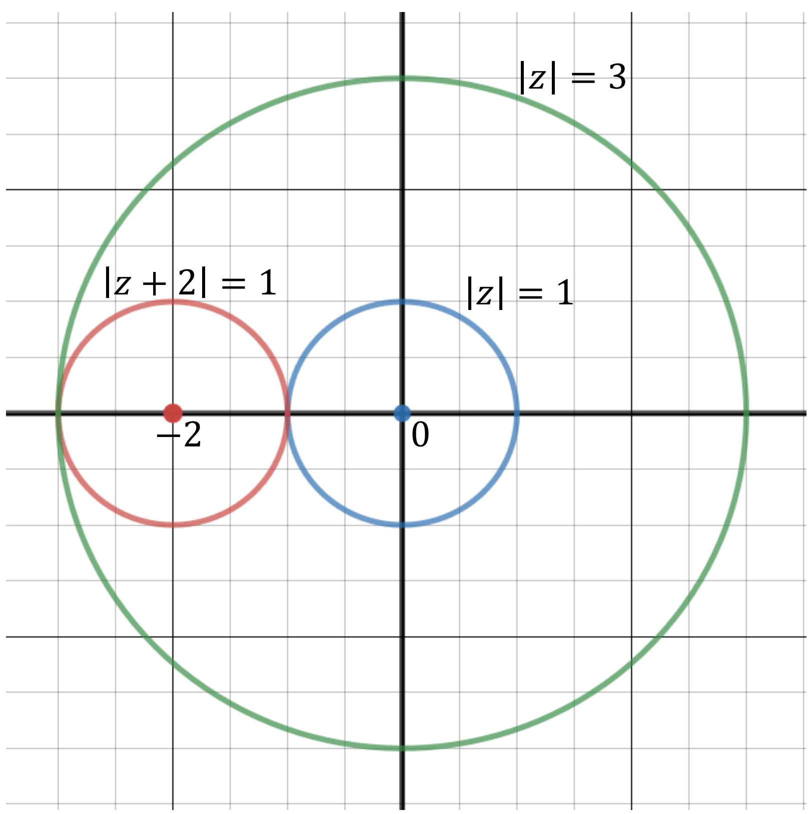
$$\frac{2z+6}{z^2+2z} = \frac{2z+6}{z(z+2)} = \frac{A}{z} + \frac{B}{z+2} = \frac{A(z+2)+Bz}{z(z+2)}$$

$$2z+6 = A(z+2) + Bz$$

At $z = 0$; $\quad 6 = 2A$; $\quad$ so $A = 3$.

At $z = -2$; $\quad 2(-2)+6 = -2B$; $\quad$ so $B = -1$.

$$\frac{2z+6}{z^2+2z} = \frac{3}{z} - \frac{1}{z+2};$$ has singularities at $z = 0, -2$.



a. The circle $|z| = 1$ encloses only the singularity at $z = 0$, thus

$$\oint_C \frac{2z+6}{z^2+2z} dz = (2\pi i)Res(f(z); 0).$$

$\frac{2z+6}{z^2+2z} = \frac{3}{z} - \frac{1}{z+2}$, around $z = 0$.

$-\frac{1}{z+2}$ is analytic so won't contribute anything to the residue of $\frac{2z+6}{z^2+2z}$ at $z = 0$.

a_{-1} for the Laurent series of $\frac{3}{z}$ around $z = 0$ is 3.

Thus $Res(f(z); 0) = 3$. So $\oint_C \frac{2z+6}{z^2+2z} dz = 2\pi i(3) = 6\pi i$.

b. The circle $|z + 2| = 1$ encloses only the singularity at $z = -2$, thus

$$\oint_C \frac{2z+6}{z^2+2z} dz = (2\pi i)Res(f(z); -2).$$

So we want the Laurent series for $f(z) = \frac{2z+6}{z^2+2z} = \frac{3}{z} - \frac{1}{z+2}$ around $z = -2$.

$\frac{3}{z}$ is analytic around $z = -2$ so won't contribute anything to the residue of $f(z)$ at $z = -2$.

a_{-1} for the Laurent series around $z = -2$, is -1 (the coefficient of the $\frac{1}{z+2}$ term). Thus

$$Res(f(z); -2) = -1 \text{ and } \oint_C \frac{2z+6}{z^2+2z} dz = 2\pi i(-1) = -2\pi i.$$

c. The circle $|z| = 3$ encloses both singularities, thus

$$\oint_C \frac{2z+6}{z^2+2z} dz = 2\pi i[Res(f(z);0) + Res(f(z);-2)]$$

$$= 2\pi i(3-1) = 4\pi i.$$

Def. Let $f(z) = \frac{h(z)}{(z-z_0)^m}$, where $h(z)$ is analytic in a neighborhood of $z = z_0$, m a positive integer, and $h(z_0) \neq 0$. We then say $f(z)$ has a **pole** of order m.

If $f(z)$ has a pole of order m then $h(z)$ is analytic near $z = z_0$ and we can write down its Taylor series:

$$h(z) = h(z_0) + h'(z_0)(z-z_0) + \cdots + \frac{h^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \cdots .$$

Now let's divide by $(z-z_0)^m$ to get an expression for $f(z)$.

$$f(z) = \frac{h(z)}{(z-z_0)^m} = \frac{h(z_0)}{(z-z_0)^m} + \frac{h'(z_0)}{(z-z_0)^{m-1}} + \cdots \frac{h^{(m-1)}(z_0)}{(m-1)!(z-z_0)} + \cdots$$

So a_{-1} for $f(z)$ is $\frac{h^{(m-1)}(z_0)}{(m-1)!}$, or (**for a pole of order m**):

$$\boldsymbol{a_{-1} = \frac{1}{(m-1)!}\frac{d^{(m-1)}}{dz^{(m-1)}}\left((z-z_0)^m f(z)\right) \quad at\ z = z_0.}$$

For a simple pole, i.e. $m = 1$, this formula becomes:

$$\boldsymbol{a_{-1} = \lim_{z \to z_0}((z-z_0)f(z)).}$$

If $f(z)$ has an **essential singularity**, i.e an isolated singularity that is not a pole of order m (e.g. $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{(2!)z^2} + \cdots + \frac{1}{(n!)z^n} + \cdots$) then calculating the Laurent series about the singularity is the only general method to calculate the residue. You can identify essential singularities by the fact that the Laurent series around the singularity will have an infinite number of terms of the form $a_{-k}(z - z_0)^{-k},\ k \in \mathbb{Z}^+$; where $a_{-k} \neq 0$.

Ex. Find the residue at $z = -2$ and $z = 0$ for $f(z) = \frac{2z+6}{z^2+2z}$ without using partial fractions or Laurent series.

$f(z) = \frac{2z+6}{z^2+2z} = \frac{2z+6}{z(z+2)}$; so $f(z)$ has a simple pole at $z = -2$ and $z = 0$.

Thus:

$$Res(f(z); -2) = a_{-1} = \lim_{z \to z_0}\left((z - z_0)f(z)\right)$$

$$= \lim_{z \to -2}(z + 2)\left(\frac{2z+6}{z(z+2)}\right)$$

$$= \lim_{z \to -2}\frac{2z+6}{z} = -1.$$

$$Res(f(z); 0) = \lim_{z \to 0}(z)\left(\frac{2z+6}{z(z+2)}\right)$$

$$= \lim_{z \to 0}\frac{2z+6}{z+2} = 3.$$

We can do the earlier example without using partial fractions or Laurent series.

Ex. Evaluate $\oint_C \frac{2z+6}{z^2+2z}dz$ where C is the circle

$a.\ |z| = 1$

$b.\ |z+2| = 1$

$c.\ |z| = 3$

$$\oint_C \frac{2z+6}{z^2+2z}dz = 2\pi i(sum\ of\ residues)$$

a. $\oint_C \frac{2z+6}{z^2+2z}dz = 2\pi i(Res(f(z);0) = 2\pi i(3) = 6\pi i$

b. $\oint_C \frac{2z+6}{z^2+2z}dz = 2\pi i(Res(f(z);-2) = 2\pi i(-1) = -2\pi i$

c. $\oint_C \frac{2z+6}{z^2+2z}dz = 2\pi i(Res(f(z);0) + Res(f(z);-2))$
$= 2\pi i(3-1) = 4\pi i.$

Ex. Find the residue at $z = 0$ of $f(z) = \frac{(\cos(z))e^z}{z^4}$.

$f(z)$ has a pole of order 4 at $z = 0$. Thus we can use the formula

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{(m-1)}}{dz^{(m-1)}}\left((z-z_0)^m f(z)\right) \quad at\ z = z_0$$

where $m = 4$ and $z_0 = 0$.

$$Res(f(z);0) = \frac{1}{3!}\frac{d^3}{dz^3}\left((z^4)\left(\frac{(\cos(z))e^z}{z^4}\right)\right)$$

$$= \frac{1}{6}\frac{d^3}{dz^3}[(\cos(z))(e^z)] \text{ evaluated at } z = 0.$$

$$\frac{d}{dz}[(\cos(z))(e^z)] = (\cos(z))(e^z) - (\sin(z))(e^z)$$

$$\frac{d^2}{dz^2}[(\cos(z))(e^z)] = (\cos(z))(e^z) - (\sin(z))(e^z) - (\sin(z))(e^z)$$
$$-(\cos(z))(e^z)$$

$$= -2(\sin(z))(e^z)$$

$$\frac{d^3}{dz^3}[(\cos(z))(e^z)] = -2(\sin(z))(e^z) - 2(\cos(z))(e^z)$$

At $z = 0$, $\frac{d^3}{dz^3}[(\cos(z))(e^z)] = -2$;

So we have: $Res(f(z);0) = \frac{1}{6}(-2) = -\frac{1}{3}$.

In the case when $f(z) = \frac{N(z)}{D(z)}$, where $N(z)$ and $D(z)$ are analytic functions, and $D(z)$ has a simple zero at $z = z_0$, while $N(z_0) \neq 0$ (thus $f(z)$ has a simple pole at $z = z_0$),

$$\boldsymbol{a_{-1} = Res(f(z); z_0) = \lim_{z \to z_0}\left[(z - z_0)\big(f(z)\big)\right] = \frac{N(z_0)}{D'(z_0)}}\,.$$

This can be useful because sometimes it's easier to calculate $\frac{N(z_0)}{D'(z_0)}$ than $\lim_{z \to z_0}\left[(z - z_0)\big(f(z)\big)\right]$. For example, to calculate the residues of $f(z) = \frac{1}{z^4+1}$, at the simple poles $z = e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}$, it's easier to use this method. We'll do this calculation in the next section.

We can see how to get this formula by using the Taylor series for $N(z)$ and $D(z)$ near $z = z_0$.

$$N(z) = N(z_0) + N'(z_0)(z - z_0) + \frac{1}{2}N''(z_0)(z - z_0)^2 + \cdots$$

$$D(z) = \qquad\qquad D'(z_0)(z - z_0) + \frac{1}{2}D''(z_0)(z - z_0)^2 + \cdots$$

So $a_{-1} = \lim_{z \to z_0}(z - z_0)\frac{N(z)}{D(z)}$

$$= \lim_{z \to z_0}(z - z_0)\frac{N(z_0)+N'(z_0)(z-z_0)+\frac{1}{2}N''(z_0)(z-z_0)^2+\cdots}{D'(z_0)(z-z_0)+\frac{1}{2}D''(z_0)(z-z_0)^2+\cdots}$$

$$= \frac{N(z_0)}{D'(z_0)}\,.$$

Ex. Find the residue at $z = 0$ for $f(z) = \csc(z)$.

$\csc(z) = \frac{1}{\sin(z)}$ has a simple pole at $z = 0$ because:

$$= \frac{1}{z - \frac{z^3}{3!} + \cdots} = \frac{1}{z(1 - \frac{z^2}{3!} + \cdots)}.$$

So $a_{-1} = Res(f(z); 0) = \frac{N(0)}{D'(0)}$;

where $N(z) = 1$, $D(z) = \sin(z)$, $D'(z) = \cos(z)$

$$a_{-1} = Res(f(z); 0) = \frac{N(0)}{D'(0)} = \frac{1}{1} = 1.$$

Ex. Evaluate $\frac{1}{2\pi i} \oint_C \frac{3z+1}{z(z-1)^3} dz$; where C is the circle $|z|$=3.

$f(z) = \frac{3z+1}{z(z-1)^3}$ has a simple pole at $z = 0$, and a pole of order 3 at $z = 1$.

Both poles are inside the circle $|z|$=3.

$$a_{-1} = Res(f(z); 0) = \lim_{z \to 0} z\left(\frac{3z+1}{z(z-1)^3}\right) = \lim_{z \to 0} \frac{3z+1}{(z-1)^3} = -1$$

$$a_{-1} = Res(f(z); 1) = \frac{1}{2!} \frac{d^2}{dz^2}\left((z-1)^3\left(\frac{3z+1}{z(z-1)^3}\right)\right)$$

$$= \frac{1}{2} \frac{d^2}{dz^2}\left(3 + \frac{1}{z}\right) = \frac{1}{2}\left(\frac{2}{z^3}\right) = 1/z^3$$

At $z = 1$; $Res(f(z); 1) = 1 \Longrightarrow \frac{1}{2\pi i} \oint_C \frac{3z+1}{z(z-1)^3} dz = -1 + 1 = 0.$

Ex. Evaluate $\frac{1}{2\pi i}\oint_C \csc(z)\,dz$; where C is the circle $|z| = 1$.

As we saw in an earlier example, $\csc(z) = \frac{1}{\sin(z)}$ has a simple pole at $z = 0$.

$Res(f(z);0) = \frac{N(0)}{D'(0)} = \frac{1}{\cos(0)} = 1$. Thus $\frac{1}{2\pi i}\oint_C \csc(z)\,dz = 1$.

Ex. Evaluate $\frac{1}{2\pi i}\oint_C z^2 \sinh\left(\frac{1}{z}\right) dz$; where C is the circle $|z| = 1$.

$\sinh\left(\frac{1}{z}\right)$ has an essential singularity (i.e. a pole of order ∞) at $z = 0$.

Thus we need to use its Laurent series to calculate the residue of $z^2 \sinh\left(\frac{1}{z}\right)$.

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \frac{1}{2}[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)]$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots$$

$$\sinh\left(\frac{1}{z}\right) = \frac{1}{z} + \frac{1}{(3!)(z^3)} + \frac{1}{(5!)(z^5)} + \cdots + \frac{1}{\left((2n+1)!\right)(z^{2n+1})} + \cdots$$

$$z^2\sinh\left(\frac{1}{z}\right) = (z^2)\left(\frac{1}{z} + \frac{1}{(3!)(z^3)} + \frac{1}{(5!)(z^5)} + \cdots + \frac{1}{((2n+1)!)(z^{2n+1})} + \cdots\right.$$

$$= z + \frac{1}{(3!)z} + \frac{1}{(5!)z^3} + \cdots + \frac{1}{\left((2n+1)!\right)(z^{2n-1})}.$$

Thus $a_{-1} = Res(f(z);0) = \frac{1}{3!} = \frac{1}{6} \Rightarrow \frac{1}{2\pi i}\oint_C z^2 \sinh\left(\frac{1}{z}\right) dz = \frac{1}{6}$