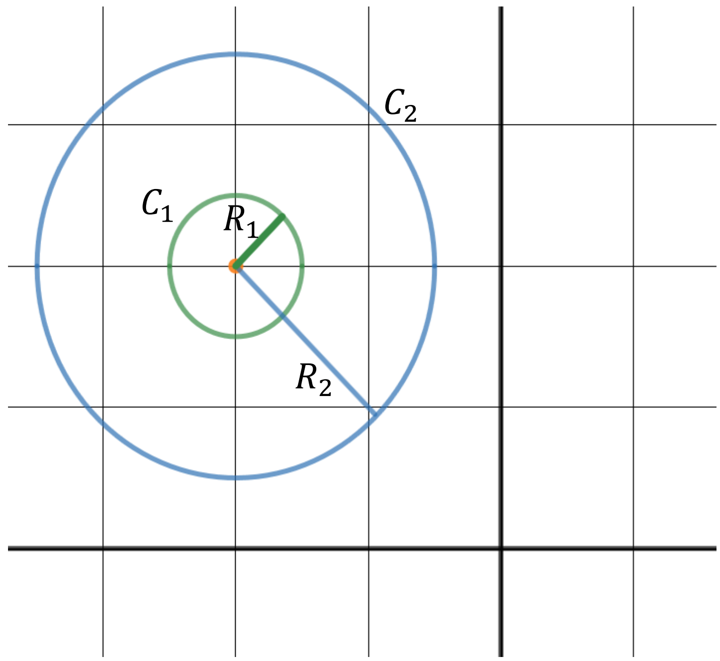
Laurent Series

Taylor series gives us a way to express an analytic function as a power series about a point $z = z_0$. However, not every function is analytic. If a function is analytic except at a finite number of points we can express the function as a sum of both positive and negative powers of $z - z_0$. We will be able to do this for functions that are analytic in and on an annulus, $R_1 \leq |z - z_0| \leq R_2$. This series is called a Laurent Series.

Note: R_1 could be 0 and/or R_2 could be ∞.



Theorem (Laurent Series) A function $f(z)$ analytic in an annulus

$R_1 \leq |z - z_0| \leq R_2$ (where R_1 could be 0 and/or R_2 could be ∞) may be represented by:

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n (z - z_0)^n$$

In a region $R_1 < R_1{}' \leq |z - z_0| \leq R_2{}' < R_2$, where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and C is any simple closed contour in the annulus that encloses the inner circle $|z - z_0| = R_1$.

This theorem is a consequence of Cauchy's integral formula and Cauchy's theorem.

1. The coefficient of the $\frac{1}{z-z_0}$ term is special and is called the **residue** of the function $f(z)$ at $z = z_0$.

2. The negative powers of the Laurent series are referred to as the **principal part** of $f(z)$.

3. If $f(z)$ is analytic everywhere inside C, then by Cauchy's theorem

$$a_j = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{j+1}} = 0 \text{ for } j < 0$$

   Because $f(z)(z - z_0)^k$ is analytic when k is a non-negative integer.

4. In practice, Laurent series are often calculated from related Taylor series as the coefficients in the previous theorem, represented as integrals, can be cumbersome to calculate.

5. The Laurent series converges uniformly to $f(z)$ in $R_1 < R_1{}' \leq |z - z_0| \leq R_2{}' < R_2$. Thus we can integrate and differentiate a Laurent series term by term in this region.

6. The Laurent series for a given annulus is unique. Thus if we can calculate a Laurent series for a function $f(z)$ using an appropriate Taylor series, then by uniqueness, that is the only Laurent series for $f(z)$ on a give annulus. Note: the annulus we use for the Laurent series matters. Different annuli can have different Laurent series for the same function.

Ex. Calculate the Laurent series for $f(z) = e^{\frac{1}{z}}$.

We know that $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$ converges for all $z \in \mathbb{C}$.

Thus substituting $\frac{1}{z}$ for z in this formula we get:

$$e^{\frac{1}{z}} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{1}{z}\right)^j = \sum_{j=0}^{\infty} \frac{1}{(j!)(z)^j}$$

which converges for $|z| > 0$.

Ex. Calculate the Laurent series for $f(z) = \frac{1}{1+z}$ on

a. $|z| < 1$
b. $|z| > 1$

a. We know the Taylor series for $f(z) = \frac{1}{1+z}$ for $|z| < 1$ is

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n .$$

Since $f(z)$ is analytic in this region, the Laurent series is the Taylor series.

b. For $|z| > 1$ we get the Laurent series as follows:

$$\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z}\sum_{j=0}^{\infty} \frac{(-1)^j}{z^j} = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^{j+1}} .$$

Notice that the Laurent series for $f(z) = \frac{1}{1+z}$ is different on the different annuli.

Ex. Find the Laurent series about the indicated singularity and find the integral around a unit circle centered at the singularity for:

a. $\frac{e^{2z}}{(z-1)^3}$; $z = 1$

b. $(z-3)\sin\left(\frac{1}{z+2}\right)$; $z = -2$

a. Let $u = z - 1$ to transform the singularity to $u = 0$.

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2(u+1)}}{u^3} = \frac{e^{2u+2}}{u^3} = \frac{e^2 e^{2u}}{u^3}$$

$$= \frac{e^2}{u^3}\left[1 + (2u) + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \cdots\right]$$

$$= \left[\frac{e^2}{u^3} + \frac{2e^2}{u^2} + \frac{2e^2}{u} + \frac{4e^2}{3} + \cdots + \frac{e^2 2^n}{n!}u^{n-3} + \cdots\right].$$

Now transform the answer back to z:

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \cdots + \frac{e^2 2^n}{n!}(z-1)^{n-3} + \cdots$$

$$\oint_C \frac{e^{2z}}{(z-1)^3}\,dz =$$

$$\oint_C \left(\frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \cdots + \frac{e^2 2^n}{n!}(z-1)^{n-3} + \cdots\right)dz;$$

where C is $|z-1| = 1$.

Recall that $\oint_C \ (z-a)^n dz = 0 \quad\quad if\ n \neq -1$

$$= 2\pi i \quad if\ n = -1.$$

where C is $|z-a| = R$.

Thus we have:

$\oint_C \ \frac{e^{2z}}{(z-1)^3} dz = 2e^2(2\pi i) = 4e^2\pi i.$ (Also can be done by Cauchy's integral formula).

b. To find the Laurent series for $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$, make the substitution $u = z+2$ to move the singularity to $u = 0$.

$$(z-3)\sin\left(\frac{1}{z+2}\right) = (u-5)\sin\left(\frac{1}{u}\right).$$

Since $\quad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$$\sin\left(\frac{1}{u}\right) = \frac{1}{u} - \frac{1}{3!(u)^3} + \frac{1}{5!(u)^5} - \frac{1}{7!(u)^7} + \cdots$$

$$(u-5)\sin\left(\frac{1}{u}\right) = (u-5)\left(\frac{1}{u} - \frac{1}{3!(u)^3} + \frac{1}{5!(u)^5} - \frac{1}{7!(u)^7} + \cdots\right)$$

$$= \left(1 - \frac{5}{u} - \frac{1}{3!(u^2)} + \frac{5}{3!(u^3)} + \cdots\right)$$

$$(z-3)\sin\left(\frac{1}{z+2}\right) = 1 - \frac{5}{z+2} - \frac{1}{3!\left((z+2)^2\right)} + \frac{5}{3!\left((z+2)^3\right)} + \cdots \quad .$$

Now integrating around the circle C: $|z+2| = 1$

$$\oint_C (z-3)\sin\left(\frac{1}{z+2}\right)dz = \oint_C \left(1 - \frac{5}{z+2} - \frac{1}{3!((z+2)^2)} + \frac{5}{3!((z+2)^3)} + \cdots\right)dz$$

Again: $\oint_C (z-a)^n dz = 0 \qquad if\ n \neq -1$

$$= 2\pi i \qquad if\ n = -1.$$

So we get:

$$\oint_C (z-3)\sin\left(\frac{1}{z+2}\right)dz = -5(2\pi i) = -10\pi i.$$

Ex. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series in powers of z valid for

a. $|z| < 1$
b. $1 < |z| < 3$
c. $|z| > 3$.

The easiest way to do this problem is with partial fractions.

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

So $1 = A(z+3) + B(z+1)$

At $z = -3$; $\quad 1 = B(-2)$ or $B = -\frac{1}{2}$

At $z = -1$; $\quad 1 = A(2)$ or $A = \frac{1}{2}$.

So we get: $\qquad \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right).$

a. If $|z| < 1$, both $\frac{1}{z+1}$ and $\frac{1}{z+3}$ are analytic so we can take their Taylor series, multiply by a constant, and subtract them.

$$\frac{1}{2}\left(\frac{1}{z+1}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n \ ;$$

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{3\left(1+\frac{z}{3}\right)}\right) = \left(\frac{1}{6}\right)\left(\frac{1}{1+\frac{z}{3}}\right)$$

$$= \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n = \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}z^n$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}z^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\left(1 - \left(\frac{1}{3}\right)\frac{1}{3^n}\right)z^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{2}(-1)^n\left(1 - \frac{1}{3^{n+1}}\right)z^n; \qquad \text{for } |z| < 1.$$

This is the Taylor series as well as the Laurent series for $\frac{1}{(z+1)(z+3)}$ since it's analytic for $|z| < 1$.

b. For $1 < |z| < 3$ we again use the partial fractions expression:

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right).$$

We need to find a Laurent (or Taylor) series that converges for each term in $1 < |z| < 3$.

For $|z| > 1$ we get the Laurent series for $\frac{1}{1+z}$ as follows:

$$\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \sum_{j=0}^{\infty}\frac{(-1)^n}{z^{n+1}}; \quad \text{Thus}$$

$$\frac{1}{2}\left(\frac{1}{z+1}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}.$$

For $|z| < 3$ we know the Taylor series for $\frac{1}{z+3}$ converges there, so from part "a" we have:

$$\frac{1}{2}\left(\frac{1}{z+3}\right) = \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}z^n$$

Thus both of these expressions converge for $1 < |z| < 3$, now subtract them.

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}z^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2)z^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2)3^{n+1}}z^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2)z^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2)3^{n+1}}z^n.$$

$\sum_{n=0}^{\infty}\frac{(-1)^n}{2z^{n+1}}$ is the principal part of the Laurent series.

c. For $|z| > 3$ we again use the partial fraction expression:

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

We need to find series that are valid for each term when $|z| > 3$.

We know that $\frac{1}{2}\left(\frac{1}{z+1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)z^{n+1}}$ is valid for $|z| > 1$, so it's also valid for $|z| > 3$.

We have to find a series that's valid for $\frac{1}{2}\left(\frac{1}{z+3}\right)$ for $|z| > 3$. Notice:

$$\frac{1}{z+3} = \frac{1}{z(1+\frac{3}{z})} = \frac{1}{z}\left(\frac{1}{1+\frac{3}{z}}\right);$$

$$= \frac{1}{z}\sum_{n=0}^{n=\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(3)^n}{z^{n+1}} \qquad \text{converges for } |z| > 3.$$

So $$\frac{1}{2}\left(\frac{1}{z+3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n(3)^n}{(2)z^{n+1}}.$$

Subtracting the two expressions we get:

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n(3)^n}{(2)z^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(1-3^n)}{(2)z^{n+1}} = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} + \cdots + \frac{(-1)^n(1-3^n)}{(2)z^{n+1}} + \cdots$$

which converges for $|z| > 3$.

Ex. Evaluate $\oint_C \frac{1}{z^3(\cos(z))}dz$; where C is the circle $|z| = 1$.

Let's find a Laurent series for $\frac{1}{z^3(\cos(z))}$.

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\frac{1}{\cos(z)} = \frac{1}{1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots} = \frac{1}{1-(\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\cdots)}$$

$$= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)^2 + \cdots$$

$$= 1 + \frac{z^2}{2} + \text{higher power terms.}$$

Note: It can be shown that if $|z| < 1$, then $|\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots| < 1$, so the series on the RHS converges.

$$\frac{1}{z^3(\cos(z))} = (\frac{1}{z^3})(1 + \frac{z^2}{2} + \text{higher powers}) = \frac{1}{z^3} + \frac{1}{2z} + \text{higher powers}$$

Now integrating around the unit circle about the origin we get:

$$\oint_C \frac{1}{z^3(\cos(z))}dz = \oint_C \left(\frac{1}{z^3} + \frac{1}{2z} + \text{higher power terms}\right)\text{dz}$$

$$= \frac{1}{2}(2\pi i) = \pi i$$

because $\oint_C z^n dz = 0 \quad if\ n \neq -1$

$$= 2\pi i \quad if\ n = -1.$$