Sequences and Series of Complex Functions

Let $f_n(z), n = 1,2,3, \dots$ be a sequence of complex functions on a region $D \subseteq \mathbb{C}$.

Ex. $f_n(z) = \dfrac{1}{nz} \qquad n = 1,2,3, \dots; \quad z \in \mathbb{C} - \{0\}$.

$$f_1(z) = \frac{1}{z}$$

$$f_2(z) = \frac{1}{2z}$$

$$f_3(z) = \frac{1}{3z}$$

$$\vdots$$

Def. We say $\boldsymbol{f_n(z)}$ **converges pointwise to** $\boldsymbol{f(z)}$ on D if $\lim\limits_{n\to\infty} f_n(z) = f(z)$ for each point $z \in D$.

This means for each $z \in D$, given $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ depending on ϵ and z, such that whenever $n \geq N$, $\quad |f_n(z) - f(z)| < \epsilon$.

If the limit doesn't exist (or is infinite) we say the sequence **diverges** for those values of z.

Ex. Prove that the $f_n(z) = \frac{1}{nz}$ converges pointwise to $f(z) = 0$ for all $z \in \mathbb{C} - \{0\}$.

To prove this we must show that for each point $z \in \mathbb{C} - \{0\}$, given any $\epsilon > 0$ there exists an $N \in \mathbb{Z}^+$ depending on ϵ and z, such that whenever $n \geq N$,

$$\left|\frac{1}{nz} - 0\right| < \epsilon.$$

We start with the ϵ statement and try to work "backwards" by solving for n to find a N that will work.

$$\left|\frac{1}{nz}\right| < \epsilon$$

$$\left|\frac{1}{n}\right| < \epsilon|z|$$

$$\frac{1}{n} < \epsilon|z| \quad \text{since } n > 0$$

$$n > \frac{1}{\epsilon|z|}.$$

Let $N > \frac{1}{\epsilon|z|}$.

Now let's show that $N > \frac{1}{\epsilon|z|}$ works, i.e. forces the ϵ statement to be true.

If $N > \frac{1}{\epsilon|z|}$ then $n \geq N$ means

$$n \geq N > \frac{1}{\epsilon|z|}$$

$$n|z| > \frac{1}{\epsilon}$$

$$\frac{1}{|nz|} < \epsilon$$

$$\left|\frac{1}{nz} - 0\right| < \epsilon.$$

Thus $f_n(z) = 1/nz$ converges pointwise to $f(z) = 0$ for all $z \in \mathbb{C} - \{0\}$.
Notice that the N we found depended on both ϵ and z.

An infinite series of function, $\sum_{j=1}^{\infty} g_j(z)$, where $g_j(z)$ is a complex function, can be viewed as a limit of an infinite sequence of partial sums $\{S_n(z)\}$, where:

$$S_n(z) = \sum_{j=1}^{n} g_j(z).$$

We say $\sum_{j=1}^{\infty} g_j(z)$ converges to a function $S(z)$ if $\lim_{n\to\infty} S_n(z) = S(z)$.

Ex. $\sum_{j=0}^{\infty} \frac{z^j}{j!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$.

In this case: $g_j(z) = \frac{z^j}{j!}$ and $S_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$.

Notice that if $\lim_{n\to\infty} S_n(z) = S(z)$, that is, the series does converge, then

$$\begin{aligned}\lim_{n\to\infty} g_n(z) &= \lim_{n\to\infty} (S_n(z) - S_{n-1}(z)) \\ &= \lim_{n\to\infty} S_n(z) - \lim_{n\to\infty} S_{n-1}(z) \\ &= S(z) - S(z) = 0.\end{aligned}$$

Thus, just as is true for the convergence of an infinite sum of real numbers, the n^{th} term must go to 0 for all values of z for which the series converges.

Def. We say a sequence of functions, $S_n(z)$, $z \in D \subseteq \mathbb{C}$, **converges uniformly** to $S(z)$ if for all $\epsilon > 0$ there exists an N (which depends just on ϵ and not z) such that if $n \geq N$, then $|S_n(z) - S(z)| < \epsilon$ for all $z \in D$.

Thus for a sequence of functions to converge uniformly in D, we must be able to find a single N that forces the ϵ statement to be true for all $z \in D$. In other words, N does NOT depend on which point z we are at.

Note: Uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

Ex. The sequence $f_n(z) = \dfrac{1}{nz}$ converges pointwise to $f(z) = 0$ for all $z \in \mathbb{C} - \{0\}$, but not uniformly.

When we showed the pointwise convergence we found that the ϵ statement

$$\left|\frac{1}{nz} - 0\right| < \epsilon$$

was eqivalent to: $n > \dfrac{1}{\epsilon|z|}$.

But for any fixed ϵ, $|z|$ can be arbitrarily close to 0 so N would need to grow toward ∞ and hence N would have to depend on z. Thus $\{f_n(z)\}$ does not converge uniformly to $f(z) = 0$.

Ex. Show that the sequence $f_n(z) = \dfrac{1}{nz}$ converges uniformly to $f(z) = 0$ in the annulus, D, $1 \leq |z| \leq 10$.

We must show given any $\epsilon > 0$ there exists an N, which depends only on ϵ (and not on the point z) such that if $n \geq N$ then

$$\left|\frac{1}{nz} - 0\right| < \epsilon.$$

Notice that in this case where $1 \leq |z| \leq 10$ the smallest $|z|$ can be is 1. That wasn't the case in our previous example where $z \in \mathbb{C} - \{0\}$.

So now if we solve the ϵ inequality we get:

$$\left|\frac{1}{nz}\right| < \epsilon$$

$$\left|\frac{1}{n}\right| < \epsilon|z|$$

$$\frac{1}{n} < \epsilon|z| \quad \text{since } n > 0$$

$$n > \frac{1}{\epsilon|z|}.$$

But since $1 \leq |z| \implies \frac{1}{\epsilon|z|} \leq \frac{1}{\epsilon}$.

So now if we choose $N > \frac{1}{\epsilon}$, the ϵ inequality should work.

(Notice that N only depends on ϵ and not z).

Now let's show that this N forces the ϵ inequality to work.

If $n \geq N > \frac{1}{\epsilon}$ then

$n > \frac{1}{\epsilon} \geq \frac{1}{\epsilon|z|}$ because $1 \leq |z|$, so

$\frac{1}{n} < \epsilon|z|$ since both sides are positive.

$\left|\frac{1}{n}\right| < \epsilon|z|$ because $n > 0$, so $\frac{1}{n} = \left|\frac{1}{n}\right|$.

$$\left|\frac{1}{nz}\right| < \epsilon \quad \implies \quad \left|\frac{1}{nz} - 0\right| < \epsilon.$$

Thus we have $f_n(z) = \frac{1}{nz}$ converges uniformly to $f(z) = 0$ in the annulus, D, $1 \le |z| \le 10$.

Theorem: let $f_n(z)$ be a sequence of continuous functions that converges uniformly to $f(z)$ in D. Then $f(z)$ is continuous and for any finite contour C inside of D:
$$\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz.$$

Proof: We must show that given any point $a \in D$, that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|z - a| < \delta$ then $|f(z) - f(a)| < \epsilon$ (here the δ can depend on the point a).

Choose any point $a \in D$, and fix an $\epsilon > 0$.

By the triangle inequality we know that:

$$|f(z) - f(a)| \le |f(z) - f_n(z)| + |f_n(z) - f_n(a)| + |f_n(a) - f(a)|.$$

Now let's show that each expression on the RHS can be made less than $\frac{\epsilon}{3}$.

Since $f_n(z)$ converges uniformly to $f(z)$ in D, we know there exists a $N \in \mathbb{Z}^+$ such that if $n \ge N$ then $|f(z) - f_n(z)| < \frac{\epsilon}{3}$ for any $z \in D$.

Thus the first and third expressions on the RHS can be made less than $\frac{\epsilon}{3}$ by choosing $n \ge N$.

Since $f_n(z)$ is continuous at $z = a$ we know there exists a $\delta > 0$ such that if $|z - a| < \delta$ then $|f(z) - f(a)| < \frac{\epsilon}{3}$. Thus with this δ

$|f(z) - f(a)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$, and $f(z)$ is continuous at $z = a$.

To show that $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$, we must show given any $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then

$$|\int_C f_n(z)dz - \int_C f(z)dz| < \epsilon.$$

But since $f_n(z)$ converges uniformly to $f(z)$ in D we know that given any $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $n \geq N$ then $|f_n(z) - f(z)| < \frac{\epsilon}{L}$, for all $z \in C$ and L =length of C.

So for $n \geq N$:

$$\begin{aligned}|\int_C f_n(z)dz - \int_C f(z)dz| &= |\int_C (f_n(z) - f(z))dz| \\ &\leq \int_C |f_n(z) - f(z)||dz| \\ &\leq \left(\frac{\epsilon}{L}\right)(L) = \epsilon.\end{aligned}$$

Hence $\lim_{n\to\infty} \int_C f_n(z)dz = \int_C f(z)dz$.

A corollary of this theorem is that if the partial sums of a series:

$$S_n(z) = \sum_{j=1}^{n} g_j(z)$$

are continuous and converge uniformly to $S(z)$ then

$$\sum_{j=1}^{\infty} \int_C g_j(z)dz = \int_C \sum_{j=1}^{\infty} g_j(z)dz = \int_C S(z)dz.$$

Def. A sequence of complex numbers $\{z_n\}$ is called **a Cauchy sequence** if for all $\epsilon > 0$, there exists a $N \in \mathbb{Z}^+$ such that if $m, n \geq N$ then $|z_n - z_m| < \epsilon$.

Theorem: If $\{f_n(z)\}$ is a Cauchy sequence for each $z \in D$, D a region in $\mathbb{C}$, then there is a function $f(z)$ such that $\{f_n(z)\}$ converges to $f(z)$.

The above theorem follows from the fact that every Cauchy sequence of real numbers converges in $\mathbb{R}$, and the following:

$$f_n(z) = u_n(x, y) + iv_n(x, y)$$

$$|u_n(x, y) - u_m(x, y)| \leq |f_n(z) - f_m(z)| < \epsilon$$

$$|v_n(x, y) - v_m(x, y)| \leq |f_n(z) - f_m(z)| < \epsilon.$$

This says that if $\{f_n(z)\}$ is a Cauchy sequence then so are $\{u_n(x, y)\}$ and $\{v_n(x, y)\}$. Hence $\{u_n(x, y)\}$ converges to $u(x, y)$ and $\{v_n(x, y)\}$ converges to $v(x, y)$.

Thus $\{f_n(z)\}$ converges to $f(z) = u(x, y) + iv(x, y)$.

The next theorem gives us a powerful tool to prove that a series of functions converges uniformly in a region $D \subseteq \mathbb{C}$

Theorem (Weierstrass M test): Let $\left|g_j(z)\right| \leq M_j$ in a region $D \subseteq \mathbb{C}$, with $M_j \in \mathbb{R}$ constants. If $\sum_{j=1}^{\infty} M_j$ converges, then the series $S(z) = \sum_{j=1}^{\infty} g_j(z)$ converges uniformly in D.

Proof: Let $n > m$ and $S_n(z) = \sum_{j=1}^{n} g_j(z)$. Then

$$|S_n(z) - S_m(z)| = \left|\sum_{j=m+1}^{n} g_j(z)\right|$$

$$\leq \sum_{j=m+1}^{n} |g_j(z)|$$

$$\leq \sum_{j=m+1}^{n} M_j \leq \sum_{j=m+1}^{\infty} M_j$$

Since $\sum_{j=1}^{\infty} M_j$ converges we know there's an $N \in \mathbb{Z}^+$ such that if $m \geq N$, $\sum_{j=m+1}^{\infty} M_j < \epsilon$.

Thus $\{S_n(z)\}$ converges uniformly to $S(z)$ in D.

Ex. Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$ converges uniformly for $|z| \leq 1$.

For $|z| \leq 1$ we have: $\left|\frac{z^n}{n\sqrt{n+1}}\right| \leq \left|\frac{1}{n\sqrt{n+1}}\right| \leq \left|\frac{1}{n^{\frac{3}{2}}}\right| = \frac{1}{n^{\frac{3}{2}}}$.

Notice that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges because it's a p-series with $p = \frac{3}{2} > 1$.

So if we let $M_j = \frac{1}{j^{\frac{3}{2}}}$ then we have:

$\left|g_j(z)\right| \leq M_j$ (here $g_j(z) = \frac{z^j}{j\sqrt{j+1}}$) and $\sum_{j=1}^{\infty} M_j$ converges.

So by the Weierstrass M test $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$ converges uniformly for $|z| \leq 1$.

Ex. Show $\sum_{n=1}^{\infty} ze^{-nz}$ converges uniformly on $1 \leq |z| \leq 4$ and $Re(z) \geq 3$.

$|ze^{-nz}| = |z||e^{-nz}| \leq 4|e^{-nz}|$ because $|z| \leq 4$.

$z = x + iy$ thus:

$4|e^{-nz}| = 4\left|e^{-n(x+iy)}\right| = 4\left|e^{-nx}e^{-nyi}\right| = 4|e^{-nx}|$;

$x = Re(z) \geq 3$; so

$|ze^{-nz}| \leq 4|e^{-nz}| = 4|e^{-nx}| \leq 4e^{-3n}$

Thus we have:

$\sum_{n=1}^{\infty} 4e^{-3n} = 4[e^{-3} + e^{-6} + e^{-9} + \cdots] = 4(\frac{e^{-3}}{1-e^{-3}})$

(Geometric series).

So if we let $M_j = 4e^{-3j}$, then $\left|g_j(z)\right| \leq M_j$ (here $g_j(z) = ze^{-jz}$) and $\sum_{j=1}^{\infty} M_j$ converges.

Thus the series $\sum_{n=1}^{\infty} ze^{-nz}$ converges uniformly on $1 \leq |z| \leq 4$ and $Re(z) \geq 3$ by the Weierstrass M test.

Ex. Show that the Riemann Zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$, converges uniformly for $Re(z) \geq 1 + \alpha$, for any $\alpha > 0$.

Let $z = x + iy$, then we have:

$$
\begin{aligned}
|n^{-z}| &= \left|e^{-zlogn}\right| \\
&= |e^{-(x+iy)logn}| \\
&= \left|e^{-xlogn}\right||e^{-iylogn}|; \qquad (\left|e^{iA}\right| = 1,\ A \in \mathbb{R}) \\
&= \left|e^{-xlogn}\right| \\
&= |n^{-x}| \\
&= \frac{1}{n^x} \leq \frac{1}{n^{1+\alpha}} = M_n.
\end{aligned}
$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}}$ which converges because it's a p-series with $p = 1 + \alpha > 1$.

Thus $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly for $Re(z) \geq 1 + \alpha$, for any $\alpha > 0$ by the Weierstrass M test.

A corollary of the Weierstrass M test is the Ratio Test for a complex series.

Theorem: Suppose $|g_1(z)|$ is bounded for $z \in D \subseteq \mathbb{C}$ and

$$\left|\frac{g_{j+1}(z)}{g_j(z)}\right| \leq M < 1, \quad j > 1$$

Then the series: $S(z) = \sum_{j=1}^{\infty} g_j(z)$ is uniformly convergent in D.

Proof: We can write: $g_n(z) = g_1(z)\left(\frac{g_2(z)}{g_1(z)}\right)\left(\frac{g_3(z)}{g_2(z)}\right) \ldots \left(\frac{g_n(z)}{g_{n-1}(z)}\right).$

Since $|g_1(z)|$ is bounded we can say: $|g_1(z)| \leq K.$

Since $\left|\frac{g_{j+1}(z)}{g_j(z)}\right| \leq M < 1, \quad j > 1$ we have

$$|g_n(z)| \leq KM^{n-1}.$$

Thus $\sum_{j=1}^{\infty}|g_j(z)| \leq K\sum_{j=1}^{\infty} M^{j-1} = \frac{K}{1-M}$ (geometric series).

So by the Weierstrass M test $\sum_{j=1}^{\infty} g_j(z)$ converges uniformly to $S(z)$ in D since

$|g_j(z)| \leq KM^{j-1} = M_j$ and $\sum_{j=1}^{\infty} M_j = \frac{K}{1-M}$ converges.

Def. The largest R for which a power series $\sum_{j=0}^{\infty} a_j z^j$ converges inside the disk $|z| < R$ is called the **radius of convergence** of the power series.

The radius of convergence of a power series can turn out to be $0, \infty$, or any positive real number.

Ex. Find the radius of convergence, R, of:

a. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, b. $\sum_{n=1}^{\infty} (n!) z^n$, c. $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^{2n}$.

By the ratio test we need to find where: $\lim_{n\to\infty} \left| \frac{g_{n+1}(z)}{g_n(z)} \right| < 1.$

a. $\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = 0 < 1$ for all $z \in \mathbb{C}$.

Thus $R = \infty$.

b. $\lim_{n\to\infty} \left| \frac{(n+1)! z^{n+1}}{(n!) z^n} \right| = \lim_{n\to\infty} |(n+1)z| = \infty$ unless $z = 0$.

Thus $R = 0$.

c. $\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} z^{2(n+1)} \frac{n!}{n^n z^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \frac{1}{n+1} z^2 \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{n^n} z^2 \right|$$

$$= \lim_{n\to\infty} \left| (\frac{n+1}{n})^n z^2 \right|$$

$$= \lim_{n\to\infty} \left| (1 + \frac{1}{n})^n z^2 \right|$$

$$= e|z^2| < 1$$

$$|z|^2 < \frac{1}{e}$$

$$|z| < \frac{1}{\sqrt{e}}$$

So $R = \frac{1}{\sqrt{e}}$.