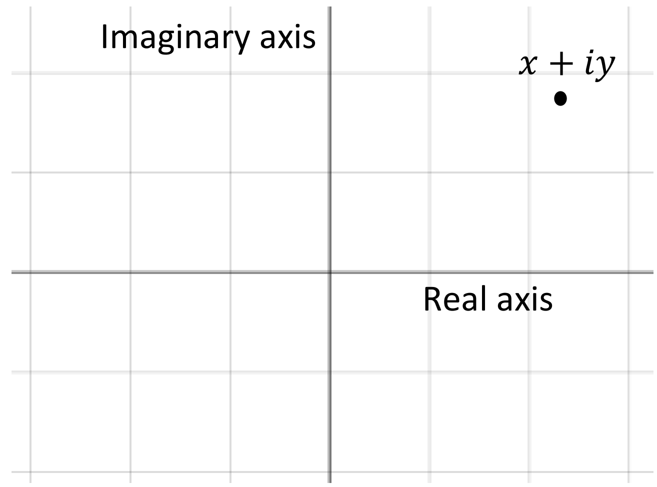
Complex Numbers

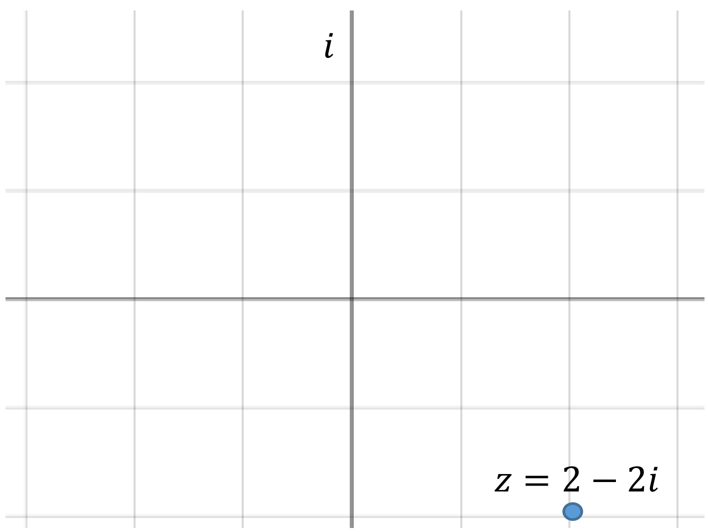
$z = x + yi$; where $i = \sqrt{-1}$

The real part of $z = Re(z) = x$

The imaginary part of $z = Im(z) = y$



Ex. $z = 2 - 2i$; $Re(z) = 2$, $Im(z) = -2$.

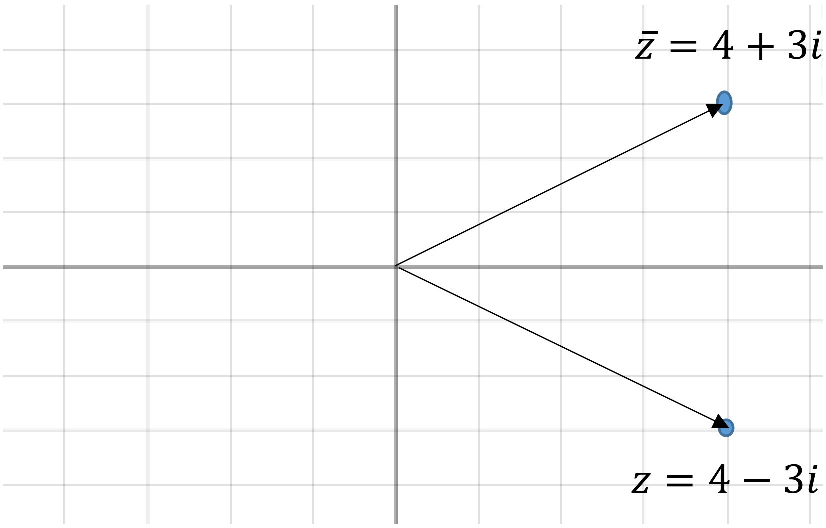


To add and subtract complex numbers we add/subtract the real parts and the imaginary parts (just like adding vectors).

Ex. $(-3 + 4i) + (2 - 2i) = -1 + 2i$.

If $z = x + yi$, then the **conjugate of z**, written $\bar{z}$, is $\bar{z} = x - yi$.

Ex. if $z = 4 - 3i$, then $\bar{z} = 4 + 3i$.



Notice that if $z = x + yi$ and $\bar{z} = x - yi$ then

$$\frac{z+\bar{z}}{2} = \frac{(x+yi)+(x-yi)}{2} = x = Re(z)$$

$$\frac{z-\bar{z}}{2i} = \frac{(x+yi)-(x-yi)}{2i} = y = Im(z).$$

It's easy to show that:

$$\overline{z+w} = \bar{z} + \bar{w}.$$

If $z = x + yi$, then $|z| = \sqrt{x^2 + y^2}$=distance from $x + yi$ to 0.

$|z|$ is called the **modulus** of z.

Notice that $|\bar{z}| = \sqrt{x^2 + (-y)^2} = |z|$.

Ex. Evaluate $|3-4i|$.

$$|3-4i| = \sqrt{3^2+(-4)^2} = 5.$$

Multiplication of complex numbers

$z = x_1 + y_1 i,$ $\quad$ $w = x_2 + y_2 i$, then

$$zw = (x_1 + y_1 i)(x_2 + y_2 i)$$

$$zw = x_1x_2 + x_1y_2 i + x_2y_1 i + y_1y_2 i^2$$

$$zw = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i \qquad \text{(since } i^2 = -1\text{)}.$$

Notice that $zw = wz$.

It is also easy to show that $|zw| = |z||w|$.

Ex. Find $(2-3i)(1+2i)$.

$$\begin{aligned}(2-3i)(1+2i) &= 2 + 4i - 3i - 6i^2 \\ &= 2 + i + 6 \\ &= 8 + i.\end{aligned}$$

Notice also that $|z|^2 = z\bar{z}$; since

$$\begin{aligned}z\bar{z} &= (x+yi)(x-yi) \\ &= x^2 - xyi + xyi - y^2i^2 \\ &= x^2 + y^2 = |z|^2.\end{aligned}$$

Division of complex numbers

$z = x_1 + y_1 i, \qquad w = x_2 + y_2 i\,,$ then

$$\frac{z}{w} = \frac{x_1+y_1 i}{x_2+y_2 i} = \left(\frac{x_1+y_1 i}{x_2+y_2 i}\right)\left(\frac{x_2-y_2 i}{x_2-y_2 i}\right)$$

$$\frac{z}{w} = \frac{(x_1+y_1 i)(x_2-y_2 i\,)}{{x_2}^2+{y_2}^2}$$

$$\frac{z}{w} = \frac{x_1x_2+y_1y_2}{{x_2}^2+{y_2}^2} + \left(\frac{x_2y_1-x_1y_2}{{x_2}^2+{y_2}^2}\right) i.$$

Ex. Put in $x + yi$ form: $\frac{2-i}{1+i}$.

$$\frac{2-i}{1+i} = \left(\frac{2-i}{1+i}\right)\left(\frac{1-i}{1-i}\right)$$

$$= \frac{2-2i-i+i^2}{1^2+1^2} = \frac{1-3i}{2} = \frac{1}{2} - \frac{3}{2}\,i.$$

Polar Coordinate form of a complex number

Recall that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots ; \qquad \text{thus we can say:}$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

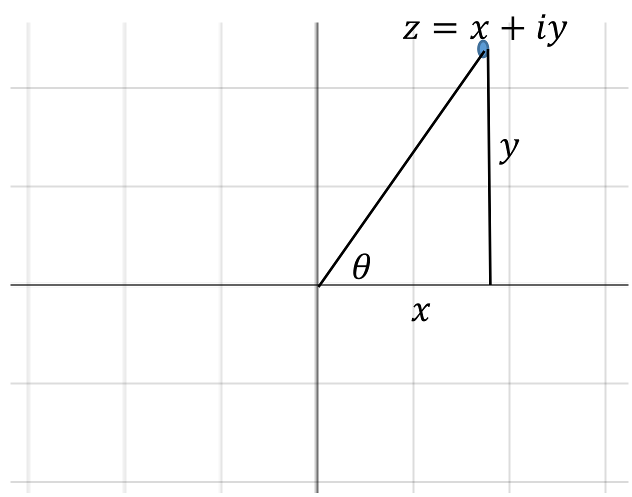
$$e^{ix} = cosx + isinx.$$

This is often called Euler's formula and is a very important relationship for the study of complex variables.

In polar coordinates we have:

$$x = rcos\theta, \qquad y = rsin\theta$$

where $r = \sqrt{x^2 + y^2}$ and $tan\theta = \frac{y}{x}$.



So $z = x + yi = rcos\theta + (rsin\theta)i$.

Since $e^{i\theta} = cos\theta + isin\theta$, we can write any complex number as:

$$z = re^{i\theta};$$

This is called the **polar form** of z.

Notice that $r = |z| =$the modulus of z.

θ is often called the **argument of** $\boldsymbol{z} = Arg(z)$.

Notice that the polar representation of a complex number is not unique. We can keep adding multiples of 2π to θ and the value of z will not change:

$$z = re^{i\theta} = re^{i(\theta+2\pi n)}, \qquad n = 0, \pm 1, \pm 2, \pm 3, \dots .$$

Multiplying and dividing complex numbers in polar form is particularly easy as is raising a complex number to a power.

$z = x_1 + y_1 i = r_1 e^{i\theta_1}, \qquad w = x_2 + y_2 i = r_2 e^{i\theta_2},$ then:

$$zw = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z}{w} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$z^n = (r_1 e^{i\theta_1})^n = (r_1{}^n) e^{in\theta_1}$$

$\bar{z} = r_1 e^{-i\theta_1}$, since $r_1 e^{-i\theta_1} = r_1 cos(-\theta_1) + r_1 \sin(-\theta_1) = x - yi = \bar{z}$.

Ex. Write $1 + i$, $\sqrt{3} + i$, and $\frac{(1+i)^3}{(\sqrt{3}+i)^2}$ in Polar form.

$z_1 = 1 + i$ so $x = 1, \ y = 1$.

Thus $r = \sqrt{1^2 + 1^2} = \sqrt{2}$;

$$tan\theta = \frac{1}{1}; \ \ so \ \ \theta = \frac{\pi}{4} + 2\pi n, \ n = 0, \pm 1, \pm 2, \dots$$

So $z_1 = \sqrt{2} e^{i(\frac{\pi}{4}+2\pi n)}$; $\quad n = 0, \pm 1, \pm 2, \dots.$

$z_2 = \sqrt{3} + i$ so $x = \sqrt{3}, \quad y = 1.$

Thus $r = \sqrt{3+1} = 2,$

$$tan\theta = \frac{1}{\sqrt{3}}; \quad so \ \ \theta = \frac{\pi}{6} + 2\pi n, \ \ n = 0, \pm 1, \pm 2, \ldots$$

So $z_2 = 2e^{i(\frac{\pi}{6}+2\pi n)}$; $\quad n = 0, \pm 1, \pm 2, \ldots.$

$$\frac{(1+i)^3}{(\sqrt{3}+i)^2} = \frac{(\sqrt{2}e^{i(\frac{\pi}{4}+2\pi n)})^3}{(2e^{i(\frac{\pi}{6}+2\pi n)})^2}$$

$$= \frac{2\sqrt{2}e^{i(\frac{3\pi}{4}+6\pi n)}}{4e^{i(\frac{\pi}{3}+4\pi n)}}$$

$$= \frac{\sqrt{2}}{2}e^{i(\frac{3\pi}{4}-\frac{\pi}{3}+2\pi n)}$$

$$= \frac{\sqrt{2}}{2}e^{i(\frac{5\pi}{12}+2\pi n)} ; \qquad n = 0, \pm 1, \pm 2, \ldots.$$

If $\quad z = re^{i(\theta+2\pi n)}, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$

then for each value of $n = 0, \pm 1, \pm 2, \ldots$, the value of z is the same. However, when we take roots (square roots, cube roots, etc.) of z this won't be the case:

If $z = re^{i(\theta+2n\pi)}$ then

$$z^{\frac{1}{m}} = (re^{i(\theta+2n\pi)})^{\frac{1}{m}} = r^{\frac{1}{m}}\left(e^{i\left(\frac{\theta}{m}+\frac{2\pi n}{m}\right)}\right); \quad n = 0, \pm 1, \pm 2, \ldots.$$

Ex. Find the roots of $z^3 = -2$.

First convert -2 to polar form: $x = -2,\ \ y = 0,$

so $r = 2, \quad tan\theta = \frac{0}{-2} = 0; \ \ so\ \ \theta = \pi + 2n\pi, \ \ n = 0, \pm 1, \pm 2, \ldots$

Thus we have: $z^3 = 2e^{i(\pi + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \ldots$

$$z = (2e^{i(\pi+2n\pi)})^{\frac{1}{3}}$$

$$z = \sqrt[3]{2}(e^{i\left(\frac{\pi}{3}+\frac{2n\pi}{3}\right)}).$$

$n = 0 \qquad z = \sqrt[3]{2}e^{i\left(\frac{\pi}{3}\right)} = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right) = \sqrt[3]{2}\,(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$

$n = 1 \qquad z = \sqrt[3]{2}e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)} = \sqrt[3]{2}\,e^{\pi i} = -\sqrt[3]{2}$

$n = 2 \qquad z = \sqrt[3]{2}e^{i\left(\frac{\pi}{3}+\frac{4\pi}{3}\right)} = \sqrt[3]{2}\,e^{\frac{5\pi i}{3}}$

$$= \sqrt[3]{2}\left(\cos\left(\frac{5\pi}{3}\right) + isin\left(\frac{5\pi}{3}\right)\right) = \sqrt[3]{2}\,(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$$

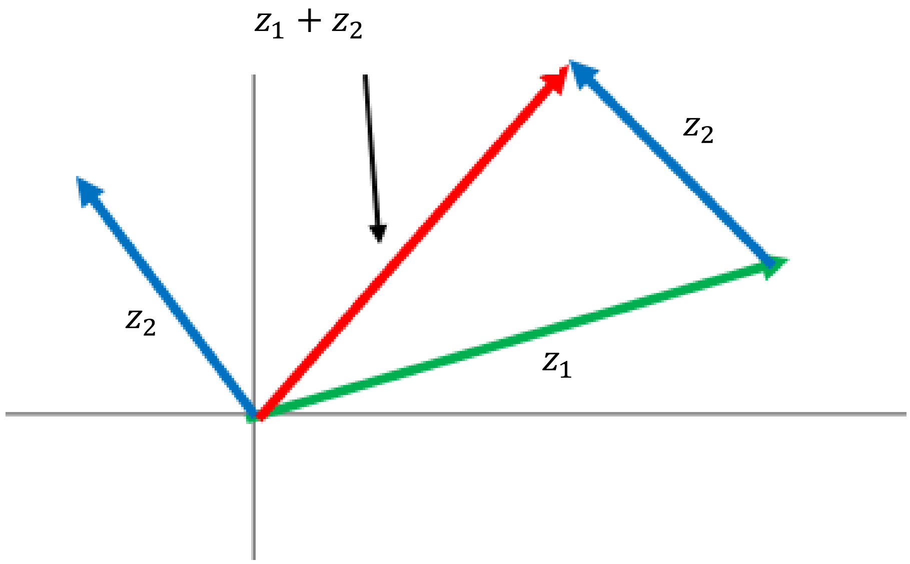
Roots: $z = \sqrt[3]{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \quad -\sqrt[3]{2}, \quad \sqrt[3]{2}\,(\frac{1}{2} - \frac{\sqrt{3}}{2}i),$ roots repeat for $n > 2$ or $n < 0.$

<u>The Triangle Inequality</u>

The triangle inequality turns out to be extremely useful in many areas of real and complex analysis.

Theorem (Triangle Inequality): For any $z_1, z_2 \in \mathbb{C}$

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$



The geometric meaning of the right half of this inequality is that the sum of any two sides of a triangle is larger than or equal to the length of the third side.

In general if $z_i \in \mathbb{C}$, then: $\left|\sum_{i=1}^{n} z_i\right| \le \sum_{i=1}^{n} |z_i|$.