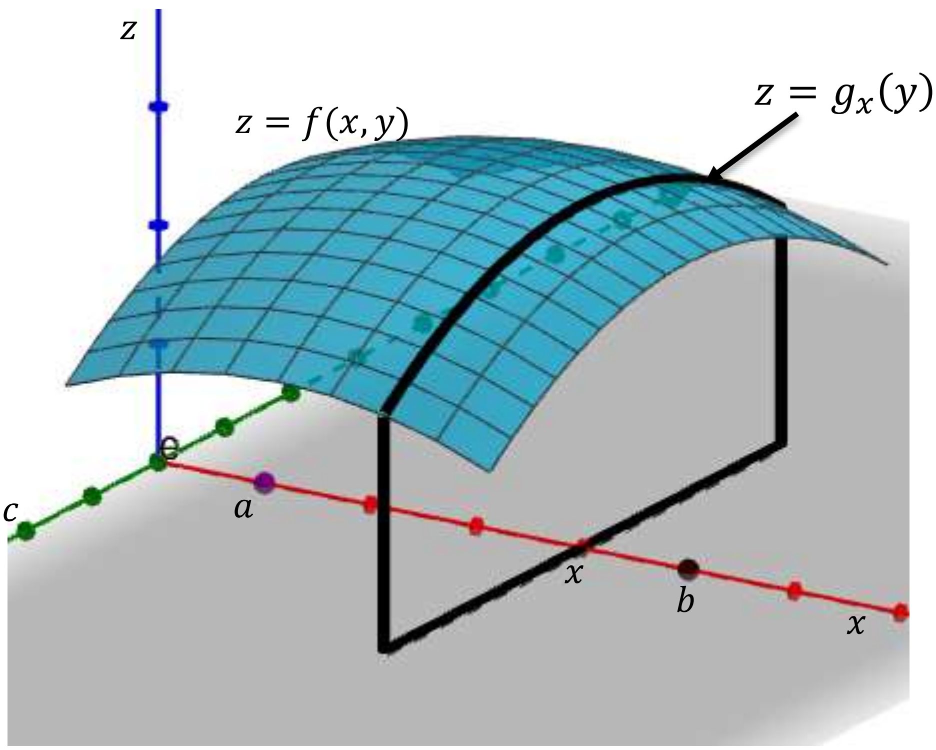
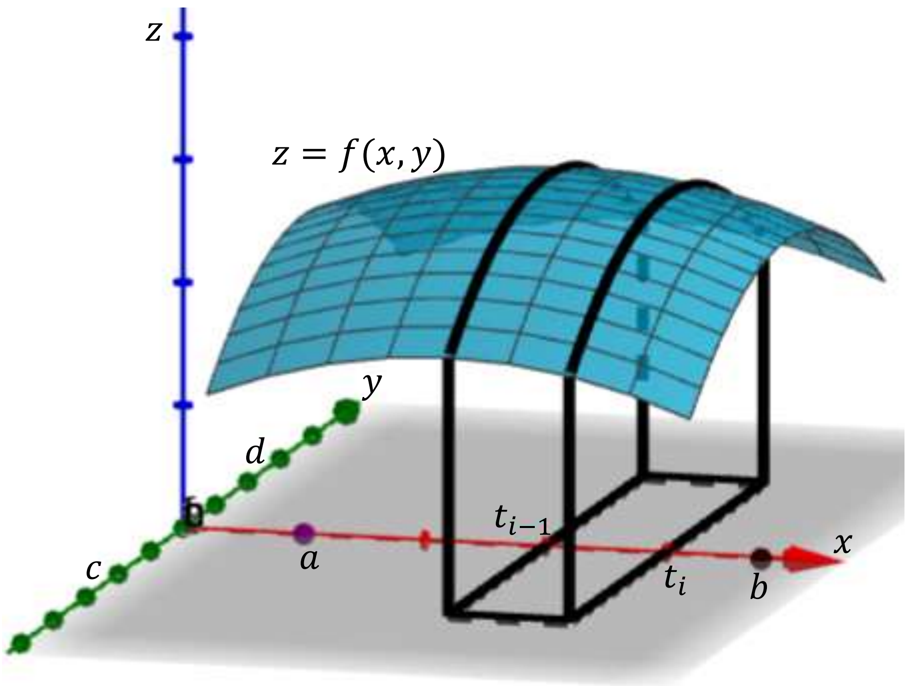
Fubini's Theorem

Suppose $f\colon [a,b] \times [c,d] \to \mathbb{R}$ is continuous and $f(x,y) \geq 0$.
Let $t_0, \ldots, t_n$ be a partition of $[a,b]$.
Define $g_x(y) = f(x,y)$ (that is, fix $x \in [a,b]$).



The area under the graph of f above $\{x\} \times [c,d]$ is:

$$\int_c^d g_x = \int_c^d f(x,y)\ dy.$$



The volume of the region under the graph of f and above $[t_{i-1}, t_i] \times [c, d]$ is approximately equal to:

$$(t_i - t_{i-1}) \int_c^d f(x, y)\ dy$$

for any $x \in [t_{i-1}, t_i]$.

Thus:

$$\int_{[a,b] \times [c,d]} f = \sum_{i=1}^{n} \int_{[t_{i-1},t_i] \times [c,d]} f$$

is approximately equal to:

$$\sum_{i=1}^{n} (t_i - t_{i-1}) \int_c^d f\ (x_i, y)\ dy$$

where $x_i \in [t_{i-1}, t_i]$.

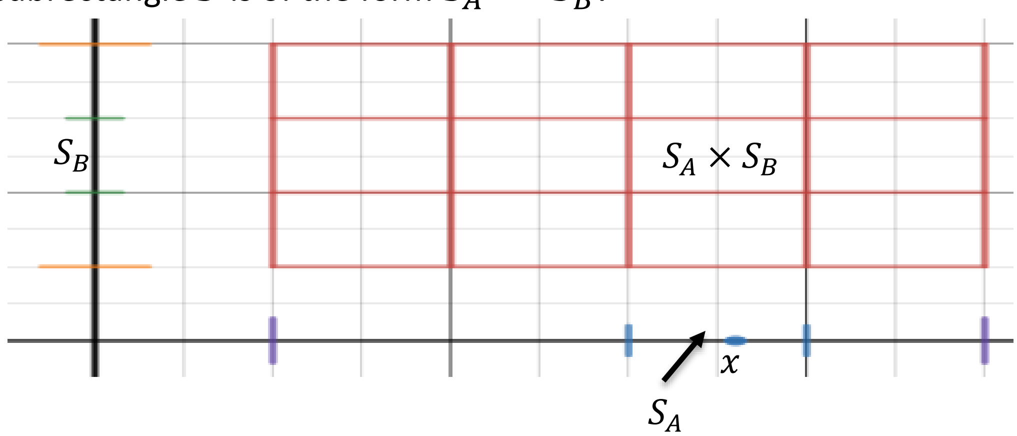
If $h(x) = \int_c^d g_x = \int_c^d f(x,y)\,dy$, then Fubini's theorem will show that h is integrable on $[a,b]$ and that:

$$\int_{[a,b]\times[c,d]} f = \int_a^b h = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

Fubini's Theorem: Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be closed rectangles, and let $f\colon A \times B \to \mathbb{R}$ be continuous (and hence integrable). For $x \in A$, let $g_x\colon B \to \mathbb{R}$ by $g_x(y) = f(x,y)$, then:

$$\int_{A\times B} f = \int_A \left[\int_B f(x,y)\,dy\right] dx.$$

Proof: Let P_A be a partition of A and P_B a partition of B.
Together P_A and P_B give a partition P of $A \times B$ where any subrectangle S is of the form $S_A \times S_B$.



$$L(f,P) = \sum_S m_S(f)\,v(S) = \sum_{S_A,S_B} m_{S_A\times S_B}(f)\,v(S_A \times S_B)$$

$$L(f,P) = \Sigma_{S_A}\left(\Sigma_{S_B} m_{S_A\times S_B}(f)\,v(S_B)\right) v(S_A).$$

If $x \in S_A$, then $m_{S_A \times S_B}(f) \le m_{S_B}(g_x)$ since $g_x(y) = f(x,y)$ and x is a fixed point in S_A. Thus we have:

$$\textstyle\sum_{S_B} m_{S_A \times S_B}(f)\, v(S_B) \le \sum_{S_B} m_{S_B}(g_x)\, v(S_B) \le \int_B g_x \,.$$

Therefore:

$$L(f,P) = \sum_{S_A} \left(\sum_{S_B} m_{S_A \times S_B}(f)\, v\,(S_B) \right) v\,(S_A)$$

$$\le \sum_{S_A} \left(\int_B g_x \right) v(S_A).$$

Since this is true for each $x \in S_A$ we can write:

$$L(f,P) \le L\left(\int_B g_x \,, P_A \right).$$

Similarly:

$$U(f,P) \ge U\left(\int_B g_x \,, P_A \right)$$

And:

$$L(f,P) \le L\left(\int_B g_x \,, P_A \right) \le U\left(\int_B g_x \,, P_A \right) \le U(f,P).$$

f is integrable so: $\sup L(f,P) = \inf U(f,P) = \int_{A\times B} f.$

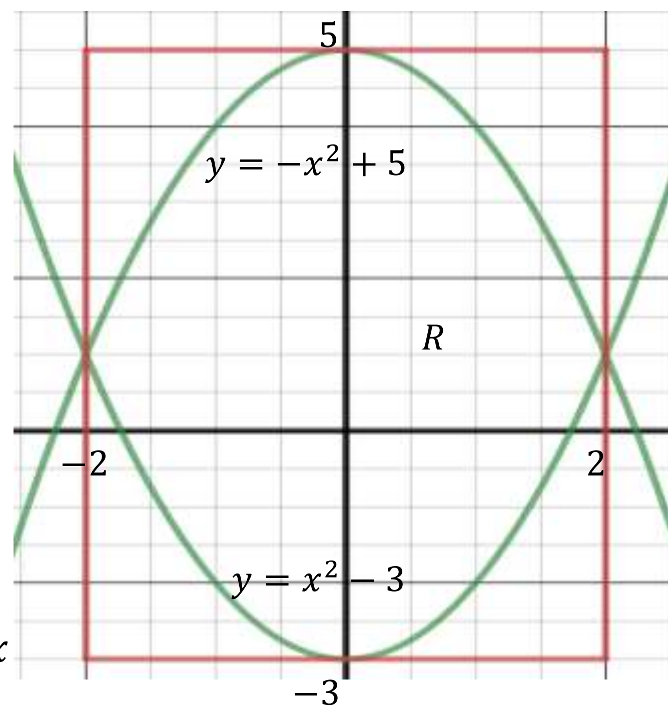
But then:

$$\sup L\left(\int_B g_x\,, P_A\right) = \inf U\left(\int_B g_x\,, P_A\right) = \int_{A\times B} f.$$

So $\int_B g_x$ is integrable on A and:

$$\int_{A\times B} f = \int_A \left[\int_B g_x\right] = \int_A \left[\int_B f(x,y)\,dy\right] dx.$$

Ex. Evaluate $\iint_R (x-2y)$ if R is the set: $x^2-3 \le y \le -x^2+5$ and $-2 \le x \le 2$.



$R \subseteq [-2,2] \times [-3,5]$ so:

$$\iint_R (x-2y) = \iint_{[-2,2]\times[-3,5]} (x-2y)\ \chi_R$$

$$= \int_{x=-2}^{x=2} \left[\int_{y=x^2-3}^{y=-x^2+5} (x-2y)dy\right] dx$$

$$= \int_{x=-2}^{x=2} xy - y^2\Big]_{y=x^2-3}^{y=-x^2+5} \, dx$$

$$= \int_{x=-2}^{x=2} [x(-x^2+5) - (-x^2+5)^2] - [x(x^2-3) - (x^2-3)^2] \, dx$$

$$= \int_{x=-2}^{x=2} -2x^3 + 8x + 4x^2 - 16 \, dx = -\frac{128}{3}.$$

Change of Variable Theorem:

Let $A \subseteq \mathbb{R}^n$ be an open set and $g: A \to \mathbb{R}^n$ a 1-1, continuously differentiable function such that $\det g'(x) \neq 0$ for all $x \in A$, except possibly for a set of measure 0.
If $f: g(A) \to \mathbb{R}$ is integrable, then:

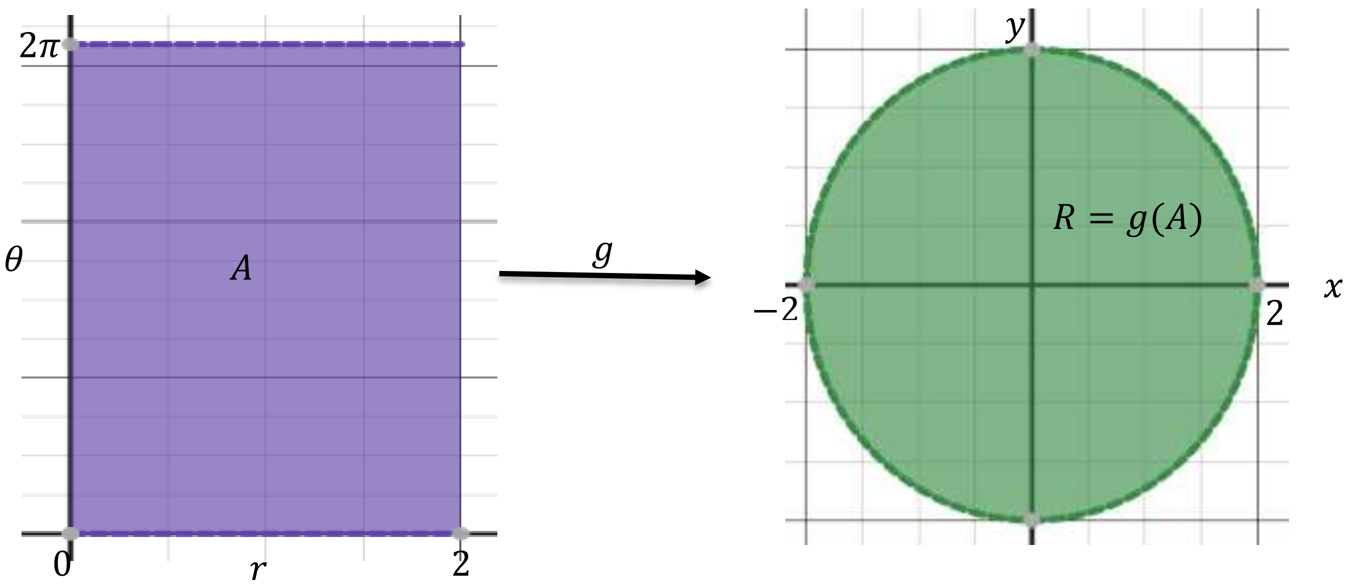
$$\int_{g(A)} f = \int_{A} (f \circ g) \, |\det g'|.$$

Ex. Evaluate $\iint_R e^{x^2+y^2}$ if $R = \{(x,y) |\, x^2 + y^2 \leq 4\}$.

Change to polar coordinates: $g(r,\theta) = (r\cos\theta, r\sin\theta)$ so $x = r\cos\theta$; $y = r\sin\theta$.

$$g' = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

So $\det g' = r\cos^2\theta + r\sin^2\theta = r$



$$f\big(g(r,\theta)\big) = e^{(r^2\cos^2\theta + r^2\sin^2\theta)} = e^{r^2}$$

$$\iint_R e^{x^2+y^2} = \iint_{[0,2]\times[0,2\pi]} e^{r^2} r = \int_{\theta=0}^{\theta=2\pi} \left[\int_{r=0}^{r=2} e^{r^2} r\, dr \right] d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}\, e^{r^2}\Big|_{r=0}^{r=2}\, d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{2}e^4 - \frac{1}{2}e^0 \right) d\theta = \pi(e^4 - 1).$$