Integration

Suppose that $A \subseteq \mathbb{R}^n$ is a closed rectangle, i.e.

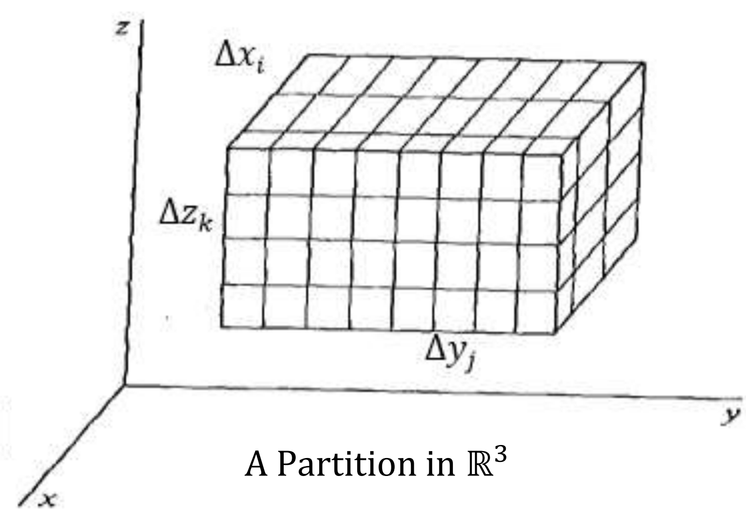
$$A = \{(x_1, \dots, x_n) |\ x_i \in [a_i, b_i], i = 1, \dots, n\}$$

We want to first discuss the definition of: $\int_A f$, where $f: A \to \mathbb{R}$.

A partition of an interval, $[a, b]$, is a sequence, $t_0, t_1, t_2, \dots, t_n$, where:

$$a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b$$

A partition of a rectangle, $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$, is a collection: $P = \{P_1, P_2, \dots, P_n\}$ where each P_i is a partition of $[a_i, b_i]$.



A Partition in $\mathbb{R}^3$

Suppose $f\colon A \to \mathbb{R}$ is a bounded function and P is a partition of A. For each subrectangle, S, of the partition let:

$$m_S(f) = \inf\{f(x) | x \in S\}$$
$$M_S(f) = \sup\{f(x) | x \in S\}$$

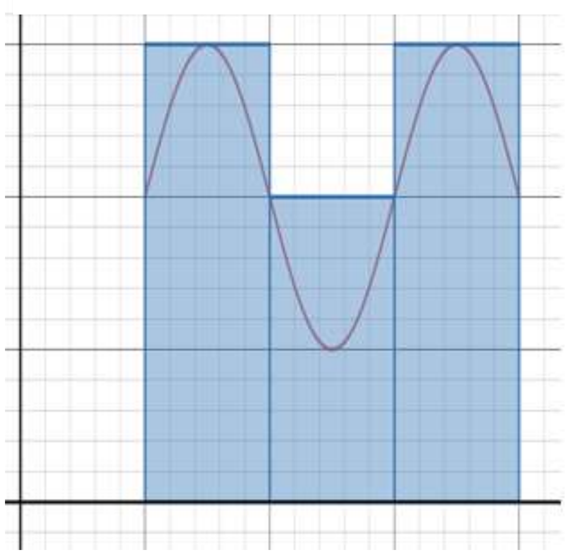
and let $v(S) =$ volume of S.

Define the **lower and upper sums** of f for P by:

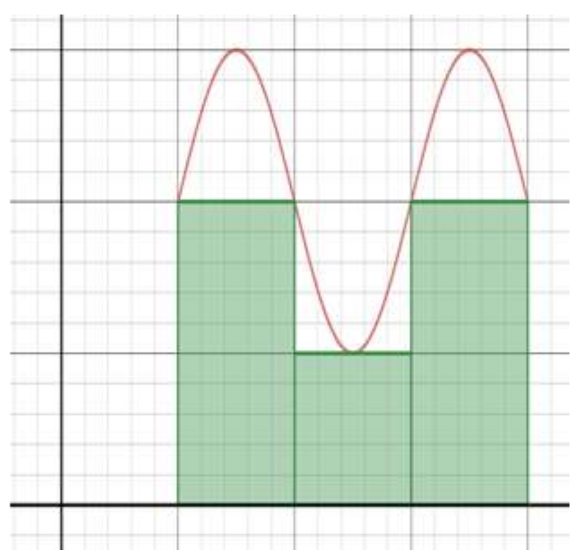
$$L(f,P) = \sum_S m_s(f) \cdot v(S)$$

$$U(f,P) = \sum_S M_S(f) \cdot v(S).$$

Upper Sum= $U(f,P)$



Lower Sum= $L(f,P)$



Notice $L(f,P) \leq U(f,P)$ since $m_s(f) \leq M_S(f)$.

Lemma: Suppose the partition P' refines P (that is each subrectangle of P' is contained in a subrectangle of P) then:

$$L(f,P) \le L(f,P') \quad ; \quad U(f,P') \le U(f,P)$$

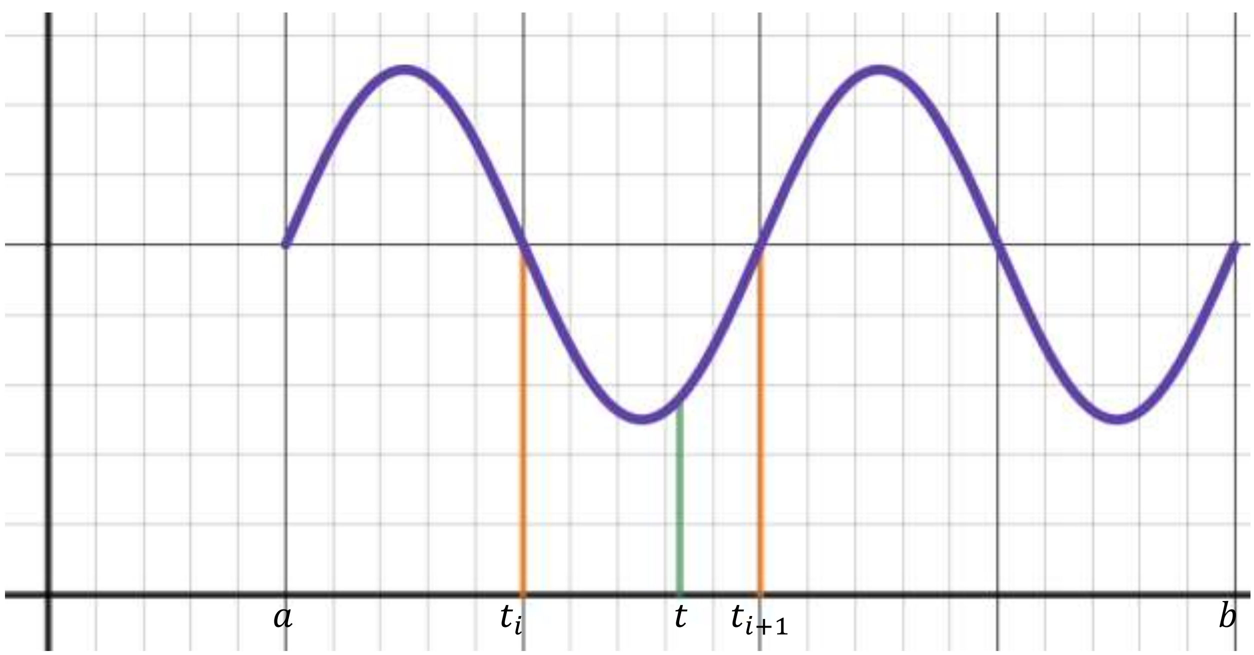
Proof (in one dimension):

$$m_i = \inf_{t_i \le x \le t_{i+1}} f(x) \qquad M_i = \sup_{t_i \le x \le t_{i+1}} f(x)$$

$$m_i' = \inf_{t_i \le x \le t} f(x) \qquad M_i' = \sup_{t_i \le x \le t} f(x)$$

$$m_i'' = \inf_{t \le x \le t_{i+1}} f(x) \qquad M_i'' = \sup_{t \le x \le t_{i+1}} f(x)$$

Notice: $m_i' \ge m_i$ and $m_i'' \ge m_i$ also $M_i' \le M_i$ and $M_i'' \le M_i$.



Thus over that interval (rectangle):

$$\begin{aligned} L(f,P') &= m_i'(t-t_i) + m_i''(t_{i+1}-t) \\ &\geq m_i(t-t_i) + m_i(t_{i+1}-t) = m_i(t_{i+1}-t_i) \\ &= L(f,P). \end{aligned}$$

$$\begin{aligned} U(f,P') &= M_i'(t-t_i) + M_i''(t_{i+1}-t) \\ &\leq M_i(t-t_i) + M_i(t_{i+1}-t) = M_i(t_{i+1}-t_i) \\ &= U(f,P). \end{aligned}$$

Corollary: If P and P' are any two partitions, then $L(f,P') \leq U(f,P)$.

Proof: Let P'' be a partition that refines P and P', then:

$$L(f,P') \leq L(f,P'') \leq U(f,P'') \leq U(f,P).$$

Def. A function, $f: A \to \mathbb{R}$, is called **integrable** if f is bounded and:

$$\sup_P(L(f,P)) = \inf_P(U(f,P)).$$

In this case: $\int_A f = \sup_P(L(f,P)) = \inf_P(U(f,P)).$

Theorem: A bounded function, $f: A \to \mathbb{R}$, is integrable if, and only if, for every $\epsilon > 0$ there is a partition P of A such that:

$$U(f,P) - L(f,P) < \epsilon$$

Proof:

If given any $\epsilon > 0$ there is a partition such that:
$U(f,P) - L(f,P) < \epsilon$, then:

$$\inf_P\big(U(f,P)\big) = \sup_P(\,L(f,P)).$$

If f is integrable, then:

$$\inf_P\big(U(f,P)\big) = \sup_P(\,L(f,P)).$$

Then for any $\epsilon > 0$ there are partitions, P and P', with:

$$U(f,P) - L(f,P') < \epsilon.$$

Let P'' be a refinement of P and P', then:

$$U(f,P'') - L(f,P'') \leq U(f,P) - L(f,P') < \epsilon.$$

Ex. Let $f: A \to \mathbb{R}$ be a constant function, $f(x) = c$. Then for any partition, P, and subrectangle, S, we have $m_S(f) = M_S(f) = c$ so we can write:

$$L(f,P) = U(f,P) = \sum_S c \,\cdot v(S) = c\big(v(A)\big)$$

Thus: $$\int_A f = c\big(v(A)\big).$$

Ex. Let $f: [0,1] \times [0,1] \to \mathbb{R}$ by:

$$f(x,y) = 0 \ ; \quad x \in \mathbb{Q}$$
$$f(x,y) = 1 \ ; \quad x \in \mathbb{R} - \mathbb{Q}$$

Show that $\int_A f$ doesn't exist.

Let P be a partition, then every subrectangle contains points, (x, y), where x is rational and points where x is irrational.

Thus: $m_S(f) = 0$, $M_S(f) = 1$, so we have

$$L(f,P) = \sum_S (0)\ v(S) = 0$$

$$U(f,P) = \sum_S (1)\ v(S) = 1$$

So $\inf U(f,P) = 1$ and $\sup L(f,P) = 0$. Thus $\int_A f$ doesn't exist.

Ex. Suppose that $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is integrable. Show that if c is a real constant, then cf is integrable and $\int_A cf = c \int_A f$.

Let's consider two cases: 1) where $c \geq 0$ and 2) where $c < 0$.

Case 1: $c \geq 0$. Let P be any partition. In any subrectangle:

$$M_S(cf) = cM_S(f)$$
$$m_S(cf) = cm_S(f)$$

Thus:

$$U(cf,P) = cU(f,P)$$
$$L(cf,P) = cL(f,P).$$

Case 2: $c < 0$.

$$M_S(cf) = -|c|m_S(f)$$
$$m_S(cf) = -|c|M_S(f)$$

Thus:

$$U(cf,P) = -|c|L(f,P)$$
$$L(cf,P) = -|c|U(f,P).$$

So if $c \geq 0$, then:

$$U(cf,P) - L(cf,P) = c\big(U(f,P) - L(f,P)\big).$$

If $c < 0$, then:

$$U(cf,P) - L(cf,P) = -|c|L(f,P) — (-|c|U(f,P))$$
$$= |c|\big(U(f,P) - L(f,P)\big).$$

Since f is integrable for all $\epsilon > 0$, there exists a partition, P, such that:

$$U(f,P) - L(f,P) < \frac{\epsilon}{|c|}.$$

Thus, for that partition:

$$U(cf,P) - L(cf,P) = |c|\big(U(f,P) - L(f,P)\big) < \frac{|c|\epsilon}{|c|} = \epsilon$$

and cf is integrable.

Since cf is integrable, we have:

$$c \geq 0 \qquad \int_A cf = \inf_{\mathrm{P}} U(cf,P) = \inf_{\mathrm{P}} cU(f,P)$$

$$= c \inf_{\mathrm{P}} U(f,P) = c\int_A f$$

$$c < 0 \qquad \int_A cf = \sup_{\mathrm{P}} L(cf,P) = \sup_{\mathrm{P}} \big(-|c|U(f,P)\big)$$

$$= -|c| \inf U(f,P) = c \inf U(f,P) \text{ since } c < 0$$

$$= c \int_A f.$$

Ex. Show that if f and g are integrable on A, then $f + g$ is integrable on A and:

$$\int_A (f+g) = \int_A f + \int_A g.$$

Notice that for any rectangle S:

$$\inf_S (f+g) \geq \inf_S (f) + \inf_S (g)$$
$$\sup_S (f+g) \leq \sup_S (f) + \sup_S (g).$$

Thus: $\quad m_S(f) + m_S(g) \leq m_S(f+g)$

$$M_S(f) + M_S(g) \geq M_S(f+g).$$

Hence we have for every partition P:

$$L(f,P) + L(g,P) \leq L(f+g,P)$$

$$U(f+g,P) \leq U(f,P) + U(g,P).$$

Thus for any partition P:

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P).$$

Since f is integrable we know given any $\epsilon > 0$ there exists a partition, Q, such that:

$$U(f,Q) - L(f,Q) < \frac{\epsilon}{2}.$$

Since g is integrable we know there exists a partition, Q', such that:

$$U(g,Q') - L(g,Q') < \frac{\epsilon}{2}.$$

Let P' be a refinement of Q and Q' thus,

$$U(f,P') - L(f,P') < \frac{\epsilon}{2}$$

$$U(g,P') - L(g,P') < \frac{\epsilon}{2}$$

$$U(f,P') + U(g,P') - \big(L(f,P') + L(g,P')\big) < \epsilon.$$

But:

$$U(f+g,P') - L(f+g,P') < U\left(f,P'\right) + U\left(g,P'\right) - \left(L\left(f,P'\right) + L\left(g,P'\right)\right) < \epsilon.$$

So $f+g$ is integrable.

Now let's show:

$$\int_A (f+g) = \int_A f + \int_A g.$$

We just saw that for partition P':

$$U(f,P') - L(f,P') < \frac{\epsilon}{2}$$

$$U(g,P') - L(g,P') < \frac{\epsilon}{2}.$$

So:

$$U(f,P') < L(f,P') + \frac{\epsilon}{2} \leq \int_A f + \frac{\epsilon}{2}$$

$$U(g,P') < L(g,P') + \frac{\epsilon}{2} \leq \int_A g + \frac{\epsilon}{2}.$$

Thus:

$$\int_A (f+g) \leq U(f+g,P') \leq U(f,P') + U(g,P') \leq \int_A f + \int_A g + \epsilon.$$

Since ϵ was arbitrary:

$$\int_A (f+g) \leq \int_A f + \int_A g.$$

If we replace f, g with $-f, -g$ in the previous inequality we get:

$$\int_A (f+g) \geq \int_A f + \int_A g.$$

Hence:

$$\int_A (f+g) = \int_A f + \int_A g.$$

Ex. Suppose that $f: A \to \mathbb{R}$ is a bounded function such that $f(x) = 0$ except at a finite number of points. Prove f is integrable and $\int_A f = 0$.

Suppose that $f(x) = 0$ except at the points $p_1, \dots, p_k$, and $f(p_i) = a_i$. Let $\epsilon > 0$ be given and let's show that we can find a partition P such that:

$$U(f,P) - L(f,P) < \epsilon.$$

Given any partition Q we can always find a refinement of Q, P, such that no two $p_i{}'s$ are in the same subrectangle and such that:

$$v(S_i) < \frac{\epsilon}{k|a_i|}.$$

Then we have: $U(f,P) - L(f,P) = \sum_S (M_S - m_S)v(S)$.

But

$$\begin{aligned} M_{S_i} - m_{S_i} &= a_i \quad \text{if } a_i \geq 0 \\ &= -a_i \quad \text{if } a_i < 0. \end{aligned}$$

So

$$M_{S_i} - m_{S_i} = |a_i|.$$

Thus $$U(f,P) - L(f,P) = \sum_S (M_S - m_S)v(S)$$
$$= \sum_{S_i} |a_i| v(S_i)$$
$$\leq \sum_{S_i} |a_i| \left(\frac{\epsilon}{m|a_i|}\right) = \epsilon.$$

So f is integrable.

Now notice that: $|U(f,P)| = |\sum_{S_i} M_{S_i} v(S_i)| \leq \sum_{S_i} |a_i| \left(\frac{\epsilon}{k|a_i|}\right) = \epsilon.$

Thus for any $\epsilon > 0$, there exists a partition P such that

$$0 \leq |U(f,P)| < \epsilon,$$

thus $\inf_P |U(f,P)| = 0$ and $\int_A f = 0$.

Measure Zero

Def. A subset $A \subseteq \mathbb{R}^n$ has an n-dimensional **measure 0** if for every $\epsilon > 0$ there is a cover $\{U_1, U_2, \ldots\}$ of A by closed rectangles such that:

$$\sum_{i=1}^{\infty} v(U_i) < \epsilon\ .$$

Any finite number of points has measure 0. Since if there are n points we can cover each point with a rectangle whose volume is less than $\frac{\epsilon}{n}$.

Prop. Any countable set of points has measure 0.

Proof: Let $A = \{a_1, a_2, a_3, \dots\}$ cover a_i with a rectangle, U_i, of volume less than $\frac{\epsilon}{2^i}$. Then:

$$\sum_{i=1}^{\infty} v(U_i) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

Theorem: If $A = A_1 \cup A_2 \cup \dots$ and each A_i has measure 0, then A has measure 0.

Proof: Since A_i has measure 0, there is a cover $\{U_{i,1}, U_{i,2}, U_{i,3}, \dots\}$ of A_i by closed rectangles such that:

$$\sum_{j=1}^{\infty} v\left(U_{i,j}\right) < \frac{\epsilon}{2^i}.$$

Thus, $\{U_{i,j}\}_{i,j=1}^{\infty}$ is a cover for A and:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v\left(U_{i,j}\right) < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

So A has measure 0.

Theorem: Let A be a closed rectangle and $f: A \to \mathbb{R}$ a bounded function. Let $B = \{x | f \text{ is not continuous at } x\}$. Then f is integrable if, and only if, B is a set of measure 0.

Def. Let $C \subseteq \mathbb{R}^n$. The **characteristic function** $\boldsymbol{\chi_C}$ of C is defined by:

$$\chi_C(x) = 0 \text{ if } x \notin C$$

$$\chi_C(x) = 1 \text{ if } x \in C.$$

If $C \subseteq A$ for some rectangle A and $f: A \to \mathbb{R}$ is bounded, then $\int_C f$ is defined as:

$$\int_{\boldsymbol{C}} \boldsymbol{f} = \int_{\boldsymbol{A}} (\boldsymbol{f})(\boldsymbol{\chi_C})$$

provided that $(f)(\chi_C)$ is integrable.

Notice that if $C \subseteq A$ is a bounded set of measure 0 and f is integrable over C then

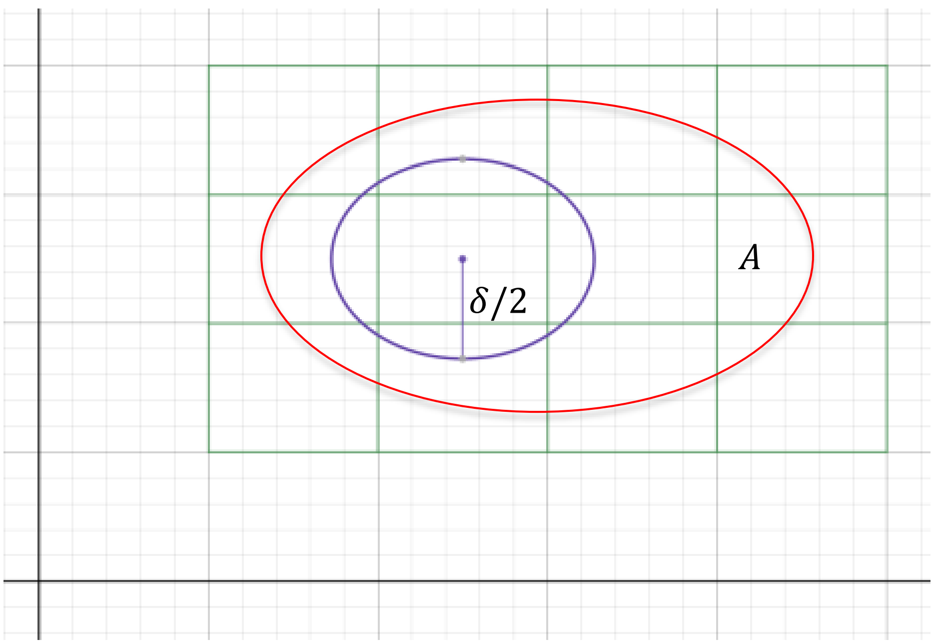
$$\int_C f = 0.$$

Prop. Suppose $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous and A is compact. Then f is integrable over A.

Proof: f is continuous on a compact set A, so f is uniformly continuous.
Thus, for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$ if

$$|x - y| < \delta \text{ then } |f(x) - f(y)| < \frac{\epsilon}{v(A)}.$$

Let P be any partition of A.
Let P' be a refinement of P such that every subrectangle S has the property that $S \subseteq B_{\frac{\delta}{2}}(\text{center of } S)$.



Thus we have for every subrectangle S:

$$(M_S - m_S) < \frac{\epsilon}{v(A)} \text{ and thus}$$

$$U(f, P') - l(f, P') = \sum_S (M_S - m_S) v(S) < \frac{\epsilon}{v(A)} v(A) = \epsilon\,.$$

So f is integrable.