Surfaces

Two common ways to represent surfaces in $\mathbb{R}^3$ are:

1) $f(x, y, z) = 0$; $x^2 + y^2 + z^2 - 1 = 0$ is the unit sphere
2) Parametrically: $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

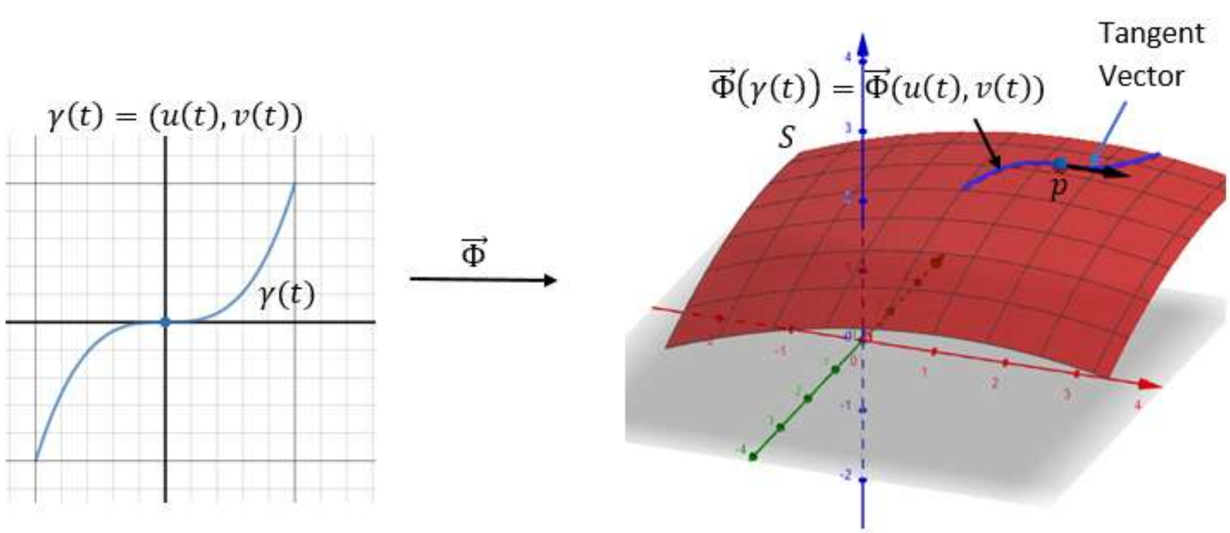
$$\vec{\Phi}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$$

e.g. $$\vec{\Phi}(u, v) = (\cos v \sin u \,, \sin v \sin u \,, \cos u)$$

where $0 \le u \le \pi \,, 0 \le v \le 2\pi$ is also a representation of the unit sphere.

If $\vec{\Phi}$ is C^1 (i.e. $x(u, v), y(u, v)$, and $z(u, v)$ have continuous partial derivatives), then $S = \vec{\Phi}(U)$ is called a C^1 surface.

Def. A **tangent vector** to a surface, $S \subseteq \mathbb{R}^3$, at a point, $p \in S$, is the velocity vector at p of a curve in S passing through p. The **tangent space**, in this case a tangent plane, of S at p is the set of all tangent vectors to S at p. We denote this by T_pS.



If $\gamma(t) = \big(u(t), v(t)\big)$ is a smooth regular curve (i.e. $\gamma'(t) \neq (0,0)$ at any point), then $\vec{\Phi}\big(u(t), v(t)\big)$ is a smooth curve on S if $\vec{\Phi}$ is smooth.

By the chain rule:

$$\frac{d}{dt}\vec{\Phi}\big(u(t), v(t)\big) = \vec{\Phi}_u\left(\frac{du}{dt}\right) + \vec{\Phi}_v\left(\frac{dv}{dt}\right)$$

where:

$$\vec{\Phi}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right); \qquad \vec{\Phi}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

$\frac{d}{dt}\vec{\Phi}\big(u(t_0), v(t_0)\big)$ is a tangent vector to S at $p = \vec{\Phi}\big(u(t_0), v(t_0)\big)$.

In fact if we take $u(t) = t, v(t) = 0$, we see that $\vec{\Phi}_u$ is a tangent vector to S at p. Similarly, if we take $u(t) = 0, v(t) = t$, we see $\vec{\Phi}_v$ is a tangent vector to S at p.

Def. $S = \vec{\Phi}(u, v)$ is called a **regular surface** at $\vec{\Phi}(u_0, v_0)$ if:

$$\vec{\Phi}_u(u_0, v_0) \times \vec{\Phi}_v(u_0, v_0) \neq \vec{0}$$

The surface is called regular if it is regular at every point.

Notice that $\vec{\Phi}_u \times \vec{\Phi}_v \neq \vec{0}$ means that $\vec{\Phi}_u$ and $\vec{\Phi}_v$ are a basis for the tangent space of S at p, because $\vec{\Phi}_u$ and $\vec{\Phi}_v$ are linearly independent so they must span the tangent plane at p.

Ex. Let $\vec{\Phi}(u,v) = (\cos v \sin u\,, \sin v \sin u\,, \cos u)$ be a parameterization of the unit sphere. Find an equation for the tangent plane at $u = \frac{\pi}{4},\ v = \frac{\pi}{3}$.

$$\vec{\Phi}_u(u,v) = (\cos v \cos u, \sin v \cos u, -\sin u)$$

$$\vec{\Phi}_u\left(\frac{\pi}{4},\frac{\pi}{3}\right) = \left(\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right), -\frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{2}\right)$$

$$\vec{\Phi}_v(u,v) = (-\sin v \sin u, \cos v \sin u, 0)$$

$$\vec{\Phi}_v\left(\frac{\pi}{4},\frac{\pi}{3}\right) = \left(\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right), \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right), 0\right) = \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, 0\right)$$

So $\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, 0\right)$ span the tangent space and their cross product is a normal vector to the tangent plane.

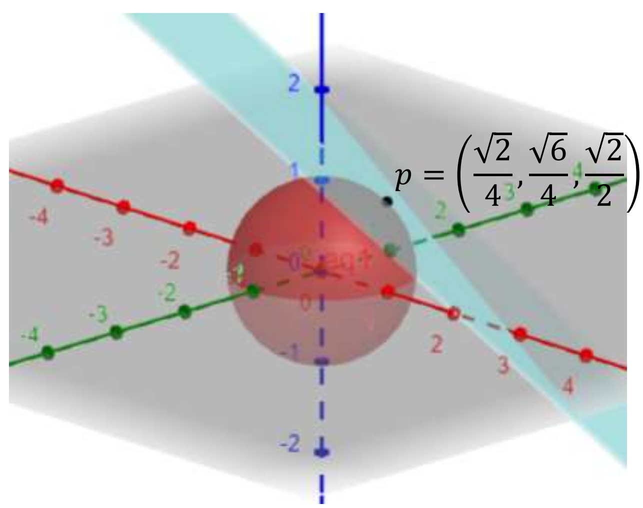
$$\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{2}\right) \times \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, 0\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & 0 \end{vmatrix} = \frac{1}{4}\vec{i} + \frac{\sqrt{3}}{4}\vec{j} + \frac{1}{2}\vec{k}.$$

The point, p, on the sphere is :

$$\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{3}\right) = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{2}\right)$$

Eq. of the tangent plane:

$$\frac{1}{4}\left(x - \frac{\sqrt{2}}{4}\right) + \frac{\sqrt{3}}{4}\left(y - \frac{\sqrt{6}}{4}\right) + \frac{1}{2}\left(z - \frac{\sqrt{2}}{2}\right) = 0.$$



We were able to find this equation easily by using the cross product, which doesn't exist in higher dimensions. However, since $\vec{\Phi}_u\left(\frac{\pi}{4},\frac{\pi}{3}\right)$ and $\vec{\Phi}_v\left(\frac{\pi}{4},\frac{\pi}{3}\right)$ span the tangent plane, we could have said:

$$T_pS = \left\{\left(\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{2}\right) + a\vec{\Phi}_u\left(\frac{\pi}{4},\frac{\pi}{3}\right) + b\vec{\Phi}_v\left(\frac{\pi}{4},\frac{\pi}{3}\right)\middle|(a,b) \in \mathbb{R}^2\right\}$$

We could express this tangent plane in another way. Notice that:

$$D\vec{\Phi} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \left(\vec{\Phi}_u\ \vec{\Phi}_v\right).$$

Since $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, then $D\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{3}\right): \mathbb{R}^2 \to \mathbb{R}^3$ and:

$$\left(D\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{3}\right)\right)\begin{pmatrix} a \\ b \end{pmatrix} = a\vec{\Phi}_u\left(\frac{\pi}{4},\frac{\pi}{3}\right) + b\vec{\Phi}_v\left(\frac{\pi}{4},\frac{\pi}{3}\right) \quad \text{for any } (a,b) \in \mathbb{R}^2.$$

Thus the tangent plane at $\left(\frac{\pi}{4},\frac{\pi}{3}\right)$ is just the image of $\mathbb{R}^2$ under $D\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{3}\right)$:

$$T_pS = \left(D\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{3}\right)\right)(\mathbb{R}^2)$$

translated to the point $\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{3}\right) = \left(\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{2}\right)$.

So if $\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$, then $D\vec{\Phi}(u_0, v_0): T_{(u_0,v_0)}U \to T_pS$.

We can represent a smooth surface in $\mathbb{R}^n$ as the image of a smooth map $\vec{\Phi}$:

$$\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^n.$$

We will still have: $\quad D\vec{\Phi}(u_0, v_0): T_{(u_0,v_0)}U \to T_pS.$

Ex. Let T^2 be the torus in $\mathbb{R}^4$ parameterized by:

$$\vec{\Phi}(u,v) = (\cos u, \sin u\,, \cos v\,, \sin v)$$

Find the tangent space of T^2 at $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ in $\mathbb{R}^4$.

$$D\vec{\Phi} = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & -\sin v \\ 0 & \cos v \end{pmatrix}.$$

At $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ we have:

$$D\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{2}\right) = \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = \left(\vec{\Phi}_u\left(\frac{\pi}{4},\frac{\pi}{3}\right) \quad \vec{\Phi}_v\left(\frac{\pi}{4},\frac{\pi}{3}\right)\right).$$

We could choose a basis for $\mathbb{R}^2$ by taking $(1,0)$ and $(0,1)$ and then take the image of the vectors under $D\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ to get $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,0\right)$ and $(0,0,-1,0)$.

Or you can just take $\vec{\Phi}_u\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ and $\vec{\Phi}_v\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ which are the same vectors.

Thus, $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,0\right)$ and $(0,0,-1,0)$ span the tangent space to T^2 at $\vec{\Phi}\left(\frac{\pi}{4},\frac{\pi}{2}\right) = \left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,1\right)$ and we have:

$$T_{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,1\right)}(T^2) = \left\{\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,1\right) + a\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0,0\right) + b(0,0,-1,0)\,\middle|\,(a,b) \in \mathbb{R}^2\right\}.$$