The Inverse Function Theorem and the Implicit Function Theorem

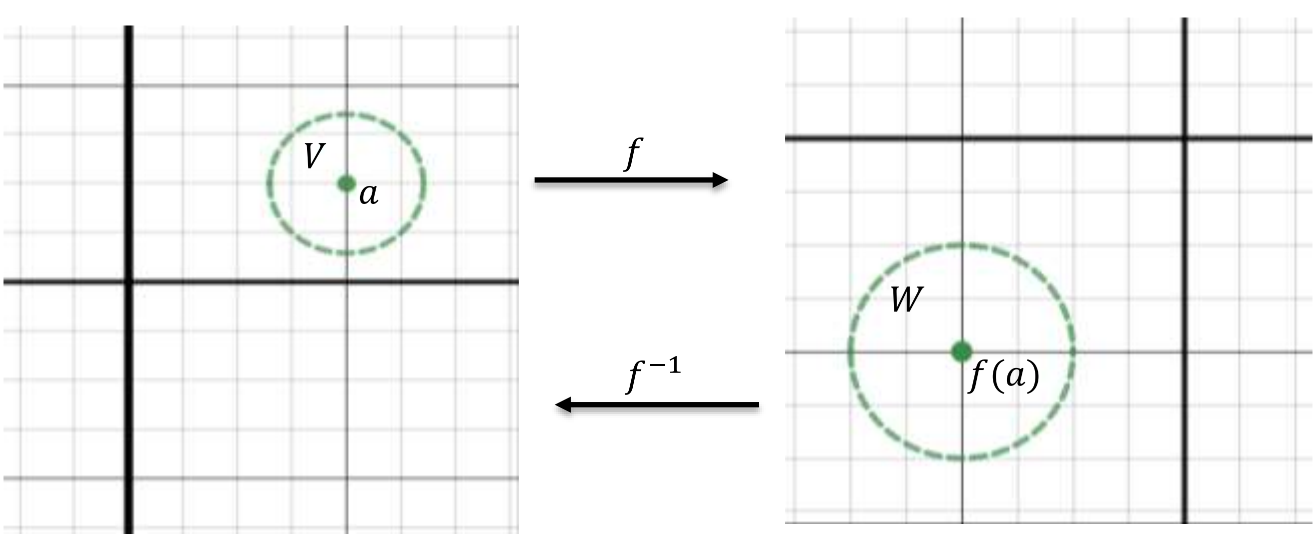
In first year calculus, we learn that if $f: \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $f'(a) \neq 0$, then there is an open interval, V, containing a such that $f'(x) > 0$ or $f'(x) < 0$ for all $x \in V$.
If $f'(x) > 0$, then f is strictly increasing on V. If $f'(x) < 0$, then f is strictly decreasing on V. Therefore, f is 1-1 on V and has an inverse function on $f(V) = W$. In addition, if $y \in W$ then,

$$(f^{-1})'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

We would like to develop a similar theorem for $f: \mathbb{R}^n \to \mathbb{R}^n$.

Inverse Function Theorem: Suppose that $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a and $\det\left(Df(a)\right) \neq 0$, then there is an open set, V, containing a and an open set, W, containing $f(a)$ such that $f: V \to W$ has a continuous inverse, $f^{-1}: W \to V$, which is continuously differentiable for $y \in W$ and satisfies:

$$(f^{-1})'(y) = \left[f'\left(f^{-1}(y)\right)\right]^{-1}.$$



Ex. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ by $F(s,t) = (s^2 - t^2, 2st)$. Show that there exists an open set, V, containing $(2,3)$ and an open set, W, containing $F(2,3) = (-5,12)$ such that F has a continuously differentiable inverse $F^{-1}: W \to V$. Find $DF^{-1}(-5,12)$ and show F does not have an inverse globally.

$$DF(s,t) = \begin{pmatrix} 2s & -2t \\ 2t & 2s \end{pmatrix}$$

so $F(s,t)$ is continuously differentiable everywhere since all of the partial derivatives are continuous everywhere.

$$DF(2,3) = \begin{pmatrix} 4 & -6 \\ 6 & 4 \end{pmatrix}$$

$$\det\big(DF(2,3)\big) = 16 + 36 = 52 \neq 0$$

So by the inverse function theorem, there exist open sets, V and W, containing $(2,3)$ and $(-5,12)$ such that $F^{-1}: W \to V$ and F^{-1} is continuously differentiable.

$$DF^{-1}(-5,12) = [DF(2,3)]^{-1}$$

$$DF^{-1}(-5,12) = \frac{1}{52}\begin{pmatrix} 4 & 6 \\ -6 & 4 \end{pmatrix}$$

$$DF^{-1}(-5,12) = \frac{1}{26}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

For F to have a global inverse, it would need to be 1-1 on all of $\mathbb{R}^2$. But $F(-1,-1) = (0,2)$ and $F(1,1) = (0,2)$, so F is not globally 1-1 and hence has no global inverse.

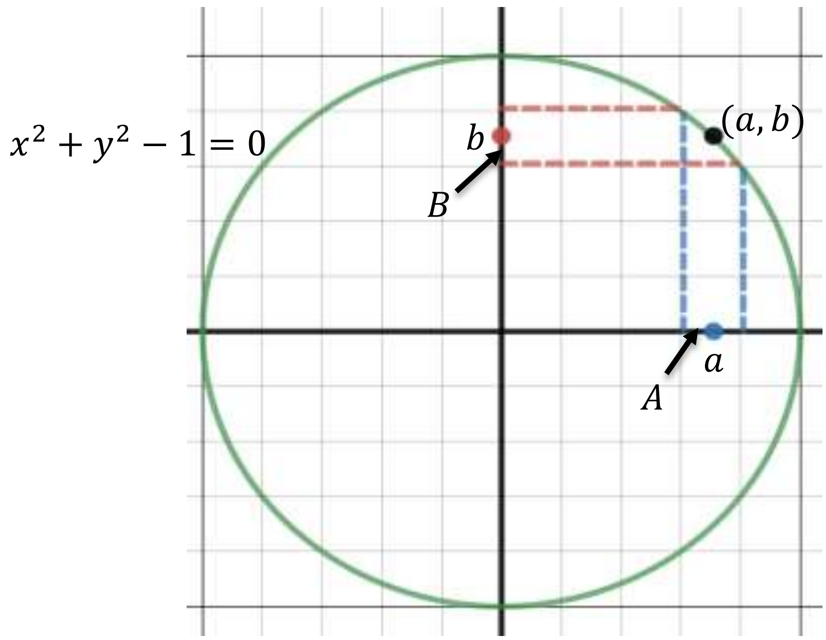
The inverse function theorem only guarantees a local inverse. In fact, f can have a local inverse at every point and not have a global inverse.

Implicit Functions

Not all relations, even in two variables, $f(x,y)=0$, can be written $y=f(x)$. Sometimes functions are defined implicitly. For example: $x^3+xy^5+y^3=0$ implicitly defines a function, y, in terms of x.

Consider $f:\mathbb{R}^2\to\mathbb{R}$ by $f(x,y)=x^2+y^2-1$.

Now let's consider all points (x,y) with $f(x,y)=0$.



Notice if we choose any point (a,b) with $f(a,b)=0, a\neq\pm1$, there are open intervals, A, containing a and B containing b such that if $x\in A$, there is a unique $y\in B$ with $f(x,y)=0$. We can define a function $g:A\to\mathbb{R}$ by $g(x)\in B$ and $f\big(x,g(x)\big)=0$, ie, we have $y=g(x)$.

Here, if $b>0$, then $g(x)=\sqrt{1-x^2}$ and if $b<0$, then $g(x)=-\sqrt{1-x^2}$. Notice that $g(x)$ is differentiable, but that when $a=\pm1$ we can't find an interval A about a where $y=g(x)$.

More generally we ask if $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $f(a_1, \dots, a_n, b) = 0$ when can we find, for each $(x_1, \dots, x_n)$ near $(a_1, \dots, a_n)$, a unique y near b, such that $f(x_1, x_2, \dots, x_n, y) = 0$ (i.e. y is implicitly a function of $(x_1, \dots, x_n)$ and $y = g(x_1, \dots, x_n)$ on A).

In fact, we can make this still more general and ask if :

$f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ by:

$$f(x_1, \dots, x_n, y_1, \dots, y_m) = (f_1(x_1, \dots, x_n, y_1, \dots, y_m), \dots, f_m(x_1, \dots, x_n, y_1, \dots, y_m))$$

with $f_i(a_1, \dots, a_n, b_1, \dots, b_m) = 0 \;\; ; \;\; i = 1, \dots, m$,

when can we find, for each $(x_1, \dots, x_n)$ near $(a_1, \dots, a_n)$ a unique $(y_1, \dots, y_m)$ near $(b_1, \dots, b_m)$ which satisfies $f_i(x_1, \dots, x_n, y_1, \dots, y_m) = 0$?

In other words, if we have:

$$\begin{aligned} &f_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ &f_2(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ &\vdots \\ &f_m(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{aligned}$$

When can we “solve” for:

$$\begin{aligned} y_1 &= g_1(x_1, \dots, x_n) \\ y_2 &= g_2(x_1, \dots, x_n) \\ &\vdots \\ y_m &= g_m(x_1, \dots, x_n). \end{aligned}$$

Implicit Function Theorem: Suppose $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable in an open set containing (a,b) and $f(a,b) = 0$ $(a \in \mathbb{R}^n,\ \ b \in \mathbb{R}^m)$. Let M be the $m \times m$ matrix:

$$M = \left(D_{n+j} f_i(a,b)\right) \qquad 1 \le i,j \le m.$$

That is: $$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{(n+1)}} \cdots & \frac{\partial f_1}{\partial x_{(n+m)}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} \cdots & \frac{\partial f_m}{\partial x_n} & \frac{\partial f_m}{\partial x_{(n+1)}} \cdots & \frac{\partial f_m}{\partial x_{(n+m)}} \end{bmatrix}$$

and $$M = \begin{bmatrix} \frac{\partial f_1}{\partial x_{(n+1)}} \cdots & \frac{\partial f_1}{\partial x_{(n+m)}} \\ \vdots & \vdots \\ \frac{\partial f_m}{\partial x_{(n+1)}} \cdots & \frac{\partial f_m}{\partial x_{(n+m)}} \end{bmatrix}.$$

If $\det M \neq 0$, then there is an open set, $A \subseteq \mathbb{R}^n$, containing a and an open set, $B \subseteq \mathbb{R}^m$, containing b where for each $x \in A$ there is a unique $g(x) \in B$, such that $f\big(x, g(x)\big) = 0$, and $g(x)$ is differentiable.

Notice that the implicit function theorem says that if we have a function, $F(x,y,z) = 0$, and $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$, then locally around$(x_0, y_0, z_0)$ the graph of $F(x,y,z) = 0$ looks like $z = g(x,y)$, where g has a differentiable inverse. Thus if $F(x,y,z) = 0$ has the property that $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$ for any point where $F(x,y,z) = 0$, then $F(x,y,z) = 0$ is a differentiable surface.

Similar statements can be made about higher dimensional objects (called manifolds). Thus, the implicit function theorem is important in differential geometry.

Ex. Let $f: \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$ (ie, $n = 3, \ m = 2$) where

$f(x_1, x_2, x_3, y_1, y_2) = (2e^{y_1} + x_1 y_2 - 4x_2 + 3, \ y_2 \cos(y_1) - 6y_1 + 2x_1 - x_3)$.

So $a = (3,2,7)$ and $b = (0,1)$ and $f(3,2,7,0,1) = (0,0)$.

$$M = \begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 2e^{y_1} & x_1 \\ -y_2 sin y_1 - 6 & cos y_1 \end{bmatrix}$$

At $(3,2,7,0,1)$ we have:

$$M = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix} \quad \text{and} \quad \det M = 20 \neq 0.$$

Thus by the implicit function theorem there exists a neighborhood $A \subseteq \mathbb{R}^3$ and $B \subseteq \mathbb{R}^2$ such that:

$$y_1 = g_1(x_1, x_2, x_3)$$
$$y_2 = g_2(x_1, x_2, x_3).$$