Partial Derivatives and Derivatives

Let $f: \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. We define the $\boldsymbol{i^{th}}$ **partial derivative of** $\boldsymbol{f}$ **at** $\boldsymbol{a}$ as

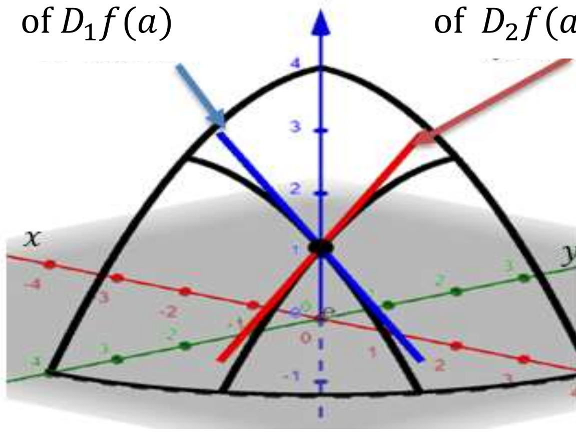
$$D_i f(a) = \lim_{h \to 0} \frac{f(a_1, a_2, \dots, a_i + h, \dots, a_n) - f(a_1, a_2, \dots, a_n)}{h}$$

as long as the limit exists.

Notice that this is just the ordinary derivative of
$g(x) = f(a_1, a_2, \dots, x, \dots, a_n)$ when $x = a_i$.

Line has slope of $D_1 f(a)$

Line has slope of $D_2 f(a)$



So we can calculate a partial derivative by holding all variables “constant” except the one we are differentiating with respect to.

Ex. Let $f(x, y, z) = \sin(x \cos y) + x^z$. Find $D_1 f = f_x,\ D_2 f = f_y,\ D_3 f = f_z$.

$$D_1 f = f_x = [\cos(x \cos y)](\cos y) + z x^{z-1}$$

$$D_2 f = f_y = [\cos(x \cos y)]\,(-x \sin y)$$

$$D_3 f = f_z = D_3\left(\left(e^{\ln x}\right)^z\right) = D_3\left(e^{z \ln x}\right)$$
$$= (\ln x) e^{z \ln x} = (\ln x)(x^z).$$

Theorem: If $D_{i,j}f$ and $D_{j,i}f$ are both continuous in an open set containing a, then

$$D_{i,j}f(a) = D_{j,i}f(a).$$

Ex. A function can have a partial derivative everywhere yet not necessarily be continuous everywhere (in contrast to the statement that if a function has a derivative at a point, then it is continuous at that point).

Let:

$$\begin{aligned} f(x,y) &= \tfrac{xy^2}{x^2+y^4} && \text{if } (x,y) \neq (0,0) \\ &= 0 && \text{if } (x,y) = (0,0) \end{aligned}$$

A direct calculation using the quotient rule shows if $(x,y) \neq (0,0)$, then:

$$f_x = \frac{y^2(y^4 - x^2)}{(x^2 + y^4)^2}$$

$$f_y = \frac{2xy(x^2 - y^4)}{(x^2 + y^4)^2}$$

If $(x,y) = (0,0)$, then:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

Thus, f_x and f_y exist everywhere.

However, $f(x,y)$ is not continuous at $(0,0)$ since

$$\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 0.$$

For:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

the limit must be 0 from all directions as $(x,y) \to (0,0)$.

If we let $x = y^2$, then we get:

$$\lim_{\substack{x=y^2\\ y\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2 \cdot y^2}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

$$\Longrightarrow \qquad \lim_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0).$$

Thus, f is not continuous at $(0,0)$.

Note: Since $f(x,y)$ is not continuous at $(0,0)$, $Df(0,0)$ does not exist.

The “problem” is that f_x and f_y are not continuous at $(0,0)$. For example, if we approach $(0,0)$ along $x = 0$ we get:

$$\lim_{\substack{y\to 0\\ x=0}} f_x = \lim_{\substack{y\to 0\\ x=0}} \frac{y^2(y^4-0)}{(0^2+y^4)^2} = \lim_{y\to 0} \frac{y^6}{y^8} = \text{undefined}$$

Derivatives:

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f_i(a)$ exists for $1 \le i \le m, 1 \le j \le n$, and $Df(a)$ is given by the $m \times n$ (Jacobian) matrix.

$$Df(a) = \left(D_j f_i(a)\right)$$

Proof: Suppose $m = 1$, so that $f: \mathbb{R}^n \to \mathbb{R}$. We define $h: \mathbb{R} \to \mathbb{R}^n$ by,

$$h(x) = (a_1, a_2, \dots, x, \dots, a_n)$$

where x is in the j^{th} place, then $(f \circ h)(x) = f(a_1, a_2, \dots, x, \dots, a_n)$ and:

$$D_j f(a) = D(f \circ h)(a_j).$$

By the chain rule the RHS becomes:

$$D_j f(a) = Df\left(h(a_j)\right) \circ Dh(a_j) = Df(a) \circ Dh(a_j)$$

But,

$$Dh(a_j) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ 1 is in the j^{th} place, so

$$D_j f(a) = Df(a) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $D_j f(a) = j^{th}$ entry of $Df(a)$.

Now for an arbitrary m; $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

We saw earlier that: $Df(a) = \big(Df_1(a), \dots, Df_m(a)\big)$, so by the first part

$Df(a) = \big(D_j f_i(a)\big)$.

Ex. Let $f(x,y) = (e^{x+2y}, \sin(x+2y))$

$g(u,v,w) = (u + 2v^2 + 3w^3, 2v - u^2)$

$h(u,v,w) = f\big(g(u,v,w)\big)$

Find $Df(x,y)$, $Dg(u,v,w)$, and $Dh(1,-1,1)$.

$$Df(x,y) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^{x+2y} & 2e^{x+2y} \\ \cos(x+2y) & 2\cos(x+2y) \end{pmatrix}$$

$$Dg(u,v,w) = \begin{pmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} & \dfrac{\partial g_1}{\partial w} \\ \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} & \dfrac{\partial g_2}{\partial w} \end{pmatrix} = \begin{pmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{pmatrix}$$

By the chain rule we know:

$$Dh(u,v,w) = Df\big(g(u,v,w)\big) \circ Dg(u,v,w).$$

When $(u,v,w) = (1,-1,1)$, $g(1,-1,1) = (1+2+3, -2-1) = (6,-3)$

$$Dh(1,-1,1) = Df(6,-3) \circ Dg(1,-1,1)$$

$$Df(6,-3) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}; \qquad Dg(1,-1,1) = \begin{pmatrix} 1 & -4 & 9 \\ -2 & 2 & 0 \end{pmatrix}$$

$$Dh(1,-1,1) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -4 & 9 \\ -2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 9 \\ -3 & 0 & 9 \end{pmatrix}.$$

Corollary: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $h: \mathbb{R}^n \to \mathbb{R}$. Then we have the following product rule for the derivative of $h(x)f(x)$:

$$D\big(h(x)f(x)\big) = \big(h(x)\big)\big(Df(x)\big) + (f(x)) \circ (Dh(x)).$$

Notice that the first term $\big(h(x)\big)\big(Df(x)\big)$ is a scalar function times an $m \times n$ matrix. Hence the result is an $m \times n$ matrix. The second term $(f(x)) \circ (Dh(x))$ is an $m \times 1$ matrix, $f(x)$, ie a column vector of dimension m, and a $1 \times n$ matrix, $Dh(x)$, ie a row vector of dimension n multipled as matrices. Thus the result is also an $m \times n$ matrix.

Proof:

$$h(x)f(x) = (h(x)f_1(x), h(x)f_2(x), \dots, h(x)f_m(x))$$

$$D\big(h(x)f(x)\big) = \begin{pmatrix} \frac{\partial(hf_1)}{\partial x_1} & \cdots & \frac{\partial(hf_1)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial(hf_m)}{\partial x_1} & \cdots & \frac{\partial(hf_m)}{\partial x_n} \end{pmatrix}$$

$$= \begin{pmatrix} h\frac{\partial(f_1)}{\partial x_1} + f_1\frac{\partial h}{\partial x_1} & \cdots & h\frac{\partial(f_1)}{\partial x_n} + f_1\frac{\partial h}{\partial x_n} \\ \vdots & \ddots & \vdots \\ h\frac{\partial(f_m)}{\partial x_1} + f_m\frac{\partial h}{\partial x_1} & \cdots & h\frac{\partial(f_m)}{\partial x_n} + f_m\frac{\partial h}{\partial x_n} \end{pmatrix}$$

$$= h\begin{pmatrix} \frac{\partial(f_1)}{\partial x_1} & \cdots & \frac{\partial(f_1)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial(f_m)}{\partial x_1} & \cdots & \frac{\partial(f_m)}{\partial x_n} \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}\begin{pmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \end{pmatrix}$$

$$= \big(h(x)\big)\big(Df(x)\big) + (f(x)) \circ (Dh(x)).$$

Ex. Let $f(x,y,z) = (xyz, x^2 + y^2 z)$ and $h(x,y,z) = 2e^{x+2y-z}$.
Find $Df(x,y,z), Dh(x,y\,,z)$, and $D(h(x,y,z)f(x,y,z))$ at the point $(x,y\,,z) = (1,1,3)$.

$$Df(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} yz & xz & xy \\ 2x & 2yz & y^2 \end{pmatrix}$$

$$Dh(x,y,z) = \begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} 2e^{x+2y-z} & 4e^{x+2y-z} & -2e^{x+2y-z} \end{pmatrix}$$

$$D(h(x,y,z)f(x,y,z)) = h(x,y,z)Df(x,y,z) + f(x,y,z) \circ Dh(x,y,z)$$

At $(1,1,3)$: $h(1,1,3) = 2\,;\;\; f(1,1,3) = (3,4)$

$$Df(1,1,3) = \begin{pmatrix} 3 & 3 & 1 \\ 2 & 6 & 1 \end{pmatrix}$$

$$Dh(1,1,3) = \begin{pmatrix} 2 & 4 & -2 \end{pmatrix}$$

$$D(hf)(1,1,3) = 2\begin{pmatrix} 3 & 3 & 1 \\ 2 & 6 & 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 2 & 4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 2 \\ 4 & 12 & 2 \end{pmatrix} + \begin{pmatrix} 6 & 12 & -6 \\ 8 & 16 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 18 & -4 \\ 12 & 28 & -6 \end{pmatrix}.$$

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f_i(x)$ exist in an open set containing a and if each function $(D_j f_i)$ is continuous at a (such a function is called **continuously differentiable**).

Proof: It's enough to prove the statement for $m = 1$, i.e. $f: \mathbb{R}^n \to \mathbb{R}$.

$$\begin{aligned} f(a+h) - f(a) = f(a_1 + h_1, a_2, \dots, a_n) - f(a_1, \dots, a_n) \\ + f(a_1 + h_1, a_2 + h_2, \dots, a_n) - f(a_1 + h_1, a_2, \dots, a_n) \\ + \cdots + f(a_1 + h_1, a_2 + h_2, \dots, a_n + h_n) - f(a_1 + h_1, \dots, a_{n-1} + h_{n-1}, a_n) \end{aligned}$$

Recall that $\frac{\partial f}{\partial x_1}(a_1, \dots, a_n) = g'(a_1)$, if $g(x) = f(x, a_2, \dots, a_n)$.

By applying the mean value theorem to $g(x)$, we get:

$$g(a_1 + h_1) - g(a_1) = h_1(g'(b_1))$$

for some b_1 between $a_1 + h_1$ and a_1.

Thus we get:

$$f(a_1 + h_1, a_2, \dots, a_n) - f(a_1, \dots, a_n) = (h_1) D_1 f(b_1, a_2, \dots, a_n).$$

Applying the same argument to the i^{th} term we get:

$$\begin{aligned} f(a_1 + h_1, \dots, a_i + h_i, a_{i+1}, \dots, a_n) - f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, a_i, \dots, a_n) \\ = (h_i) D_i f(a_1 + h_1, \dots, a_{i-1} + h_{i-1}, b_i, a_{i+1}, \dots, a_n) \end{aligned}$$

where b_i between $a_i + h_i$ and a_i.

Now we can show that $Df(a)$ equals the Jacobian matrix:

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-\sum_{i=1}^{n} D_i f(a)h_i|}{|h|}$$

$$=\lim_{h\to 0}\frac{|\sum_{i=1}^{n}(h_i)[D_i f(a_1+h_1,\dots,a_{i-1}+h_{i-1},b_i,a_{i+1},\dots,a_n)-D_i f(a)]|}{|h|}$$

$$\le \lim_{h\to 0}\sum_{i=1}^{n}|[D_i f(a_1+h_1,\dots,a_{i-1}+h_{i-1},b_i,a_{i+1},\dots,a_n)-D_i f(a)]|\frac{|h_i|}{|h|}$$

$$\le \lim_{h\to 0}\sum_{i=1}^{n}|[D_i f(a_1+h_1,\dots,a_{i-1}+h_{i-1},b_i,a_{i+1},\dots,a_n)-D_i f(a)]|$$

$= 0;$ since $D_i f$ is continuous at a.

The previous theorem says that if $f:\mathbb{R}^n\to\mathbb{R}^m$ and all of the partial derivatives of f are continuous at a then $Df(a)$ exists. However, it is possible for $Df(a)$ to exist even if all of the partial derivatives of f aren't continuous at a.

Ex. Let $f(x,y)=(x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}})$ if $(x,y)\neq(0,0)$

$= 0$ if $(x,y)=(0,0)$.

Show that $Df(0,0)$ exists but f_x is not continuous at $(0,0)$.

By the product rule we have when $(x,y)\neq(0,0)$:

$$f_x(x,y)=(x^2+y^2)\left[\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\right]\left(-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\right)+2x\sin(\frac{1}{\sqrt{x^2+y^2}})$$

$$=-\frac{x}{\sqrt{x^2+y^2}}(\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right))+2x\sin(\frac{1}{\sqrt{x^2+y^2}}).$$

Thus we have:

$$f_x(x,0) = -\frac{x}{|x|}\left(\cos\left(\frac{1}{|x|}\right)\right) + 2x(\sin\left(\frac{1}{|x|}\right)).$$

Thus $\lim_{x\to 0} f_x(x,0)$ does not exist and hence $f_x(x,y)$ is not continuous at $(0,0)$.

However, we can show that $Df(0,0)$ exists and equals $(0\ \ 0)$.

$$Df(0,0) = \lim_{h\to(0,0)} \frac{|f(0+h)-f(0)-0(h)|}{|h|}$$

$$= \lim_{h\to(0,0)} \frac{|(h_1^2+h_2^2)\left(\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)\right)|}{\sqrt{h_1^2+h_2^2}}$$

$$= \lim_{h\to(0,0)} |\sqrt{h_1^2+h_2^2}\left(\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)\right)|.$$

But:

$$0 \le \sqrt{h_1^2+h_2^2}\left(\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)\right) \le \sqrt{h_1^2+h_2^2}\,.$$

So by the squeeze theorem we have:

$$\lim_{h\to(0,0)} |\sqrt{h_1^2+h_2^2}\left(\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)\right)| = 0.$$

Thus $Df(0,0)$ exists and equals $(0\ \ 0)$.