Green's Theorem, Stokes' Theorem, the Divergence Theorem and the Fundamental Theorem of Calculus

In this section we will show that The Fundamental Theorem of Calculus, Green's Theorem, Stokes' Theorem (on surfaces in $\mathbb{R}^3$), and the Divergence Theorem are all special cases of Stokes' Theorem on manifolds.

Def. A singular 0-cube, c, is a map of $c: \{0\} \to A \subseteq \mathbb{R}^n$. If ω is a zero form (i.e. a real valued function) we define:

$$\int_{\boldsymbol{c}} \boldsymbol{\omega} = \boldsymbol{\omega}\big(\boldsymbol{c}(\boldsymbol{0})\big).$$

The Fundamental Theorem of Calculus:

If f is a smooth function on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

In this case, $df = f'(x)dx$. If we call $[a, b] = I$, then The Fundamental Theorem of Calculus becomes:

$$\int_I df = \int_{\partial I} f = f(b) - f(a).$$

That is, The Fundamental Theorem of Calculus is Stokes' Theorem where $M = I$.

Green's Theorem:

Let D be a simple region and C be its boundary. Suppose that $P: D \to \mathbb{R}$ and $Q: D \to \mathbb{R}$ are smooth function, then:

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

Notice that if $\omega = P(x,y)dx + Q(x,y)dy$, then:

$$d\omega = \frac{\partial P}{\partial y}\, dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

Since $C = \partial D$, we can write Green's Theorem as:

$$\int_{\partial D} \omega = \int_D d\omega.$$

This is Stokes' Theorem where $M = D$.

Stokes' Theorem (for parameterized surfaces in $\mathbb{R}^3$):

Let S be an oriented surface in $\mathbb{R}^3$ defined by a one-to-one parameterization $\vec{\Phi}: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, where D is a simple region. Let $F(x,y,z)$ be a smooth vector field on S, then:

$$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds.$$

To see that this is a special case of Stokes' Theorem on manifolds we must show that if $\omega = F \cdot ds$, then $d\omega = (\nabla \times F) \cdot dS$.

$$ds = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$

$$F(x,y,z) = F_1(x,y,z)\vec{\imath} + F_2(x,y,z)\vec{\jmath} + F_3(x,y,z)\vec{k}$$

$$\omega = F \cdot ds = F_1 dx + F_2 dy + F_3 dz.$$

Taking the differential of each side of the last equation we get:

$$\begin{aligned} d\omega &= d(F_1 dx + F_2 dy + F_3 dz) \\ &= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dx \\ &\quad + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dy \\ &\quad\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dz \end{aligned}$$

$$\begin{aligned} &= \frac{\partial F_1}{\partial y}dy \wedge dx + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy \\ &\quad + \frac{\partial F_2}{\partial z}dz \wedge dy + \frac{\partial F_3}{\partial x}dx \wedge dz + \frac{\partial F_3}{\partial y}dy \wedge dz \end{aligned}$$

$$d\omega = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\,dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz\,dx$$

We need to show $(\nabla \times F) \cdot dS = d\omega$.

$$\nabla \times F = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

If the surface S is given by:

$$\vec{\Phi}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big),$$

then:

$$dS = \big(\vec{\Phi}_u \times \vec{\Phi}_v\big) du\, dv$$

$$\vec{\Phi}_u \times \vec{\Phi}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}$$

$$dS = \big(\vec{\Phi}_u \times \vec{\Phi}_v\big) du\, dv = \left(\frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}\right) du\, dv.$$

Thus we can write:

$$(\nabla \times F) \cdot dS = \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\right] \cdot \left[\left(\frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}\right)\right] du\, dv$$

$$= \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\frac{\partial(z,x)}{\partial(u,v)} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)}\right] du\, dv$$

Notice that if $x = x(u,v), y = y(u,v)$, then:

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv; \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) \wedge \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du \wedge dv$$

$$= \left(\frac{\partial(x,y)}{\partial(u,v)}\right) du \wedge dv.$$

Similarly,

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)} du \wedge dv \text{ and } dz \wedge dx = \frac{\partial(z,x)}{\partial(u,v)} du \wedge dv$$

Thus we have:

$$(\nabla \times F) \cdot dS =$$

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy$$

$$= d\omega.$$

Now we can write the following two equations:

$$\iint_S (\nabla \times F) \cdot dS = \iint_S d\omega$$

$$\int_{\partial S} F \cdot ds = \int_{\partial S} \omega$$

So

$$\iint_S d\omega = \int_{\partial S} \omega.$$

Divergence Theorem:

Let $W \subseteq \mathbb{R}^3$ be an elementary region with ∂W, a closed surface given the outward orientation. Let F be a smooth vector field on W, then:

$$\iiint_W \big(div(F)\big)\, dV = \iint_{\partial W} F \cdot dS.$$

To show this is just Stokes' Theorem with $M = W$, we need to show that if $\omega = F \cdot dS$, then $d\omega = div(F)\, dV$.

We just saw in our discussion of Stokes' Theorem for surfaces that:

$$dS = (dy \wedge dz)\vec{\imath} + (dz \wedge dx)\vec{\jmath} + (dx \wedge dy)\vec{k}.$$

If $F(x, y, z) = F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k}$, then:

$$\omega = F \cdot dS = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

We need to show $d\omega = \big(div(F)\big)dx\ dy\ dz$.

$$d\omega = \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial F_3}{\partial z} dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz.$$

So if $\omega = F \cdot dS$, then $d\omega = div(F)dx\ dy\ dz$. Thus, the Divergence Theorem says:

$$\iiint_W \big(div(F)\big)dV = \iint_{\partial W} F \cdot dS$$

Or

$$\iiint_W d\omega = \iint_{\partial W} \omega.$$