Integrating Differential Forms over Manifolds

Def. If ω is a p-form on a k-dimensional manifold with boundary M and c a singular p-cube in M, then we define:

$$\int_{\boldsymbol{c}} \boldsymbol{\omega} = \int_{[\mathbf{0},\mathbf{1}]^{\boldsymbol{k}}} \boldsymbol{c}^* \boldsymbol{\omega}.$$

If c is a p-chain, then we also use the definition above.

Ex. Let T^2 be the torus embedded in $\mathbb{R}^4$ by:

$$\vec{\Phi}(u,v) = (\cos u\,, \sin u\,, \cos v\,, \sin v)\ ; \quad (u,v) \in [0,2\pi]^2$$

Let ω be given in $\mathbb{R}^4$ by $\omega = -x_2 x_3 dx_1 \wedge dx_4$.

Evaluate:

$$\int_{T^2} \omega\ .$$

$$\vec{\Phi}^*(-x_2 x_3 dx_1 \wedge dx_4) = \left(-x_2 x_3 \circ \vec{\Phi}\right) \vec{\Phi}^*(dx_1) \wedge \vec{\Phi}^*(dx_4)$$

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$$= (-\sin u \cos v)(-\sin u\, du) \wedge (\cos v\, dv)$$

$$= \sin^2 u \cos^2 v\, du \wedge dv.$$

$$\int_{T^2} \omega = \int_0^{2\pi} \int_0^{2\pi} \vec{\Phi}^*\,(\omega) = \int_0^{2\pi} \int_0^{2\pi} \sin^2 u \cos^2 v\, du\, dv$$

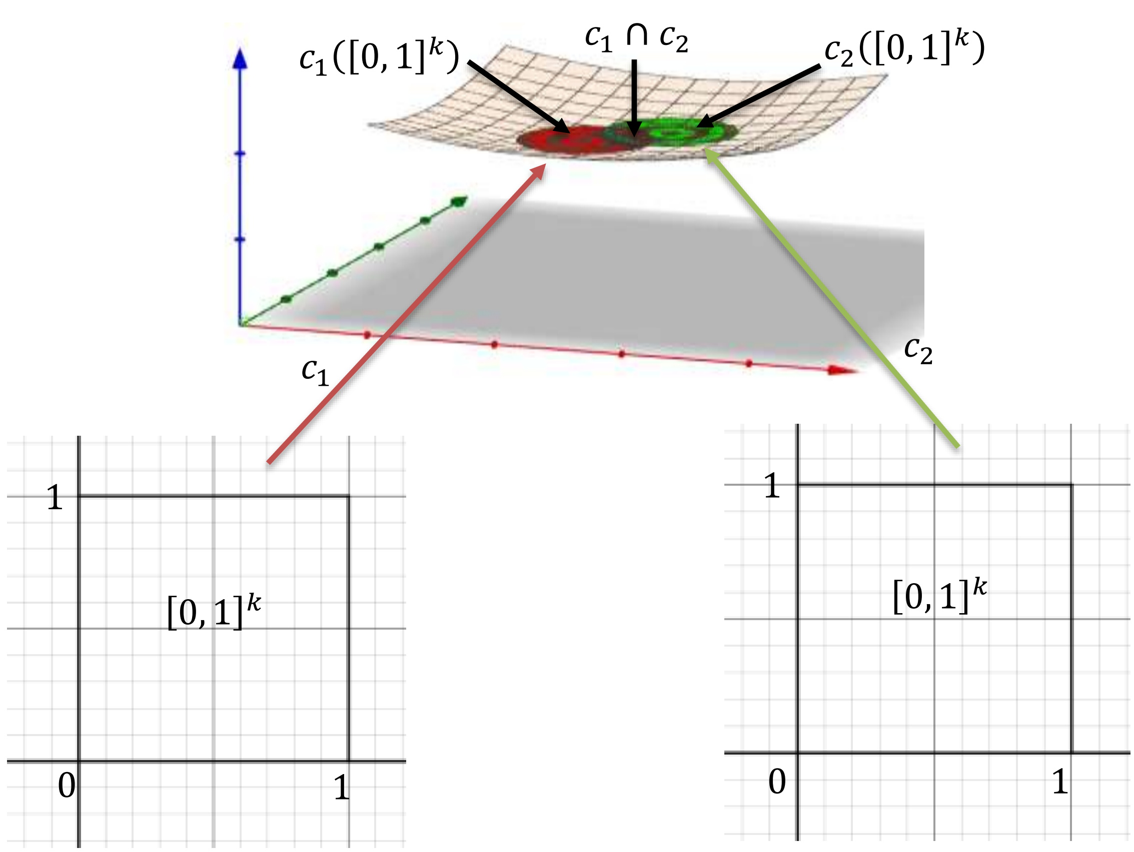
$$= \left(\int_0^{2\pi} \sin^2 u\ du\right)\left(\int_0^{2\pi} \cos^2 v\ dv\right)$$

$$= \left(\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2u\right)\, du\right)\left(\int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2v\right)\, dv\right)$$

$$= \left(\left(\tfrac{1}{2}u - \tfrac{1}{4}\sin 2u\right)\Big|_0^{2\pi}\right)\left(\left(\tfrac{1}{2}v + \tfrac{1}{4}\sin 2v\right)\Big|_0^{2\pi}\right) = \pi^2.$$

Theorem: If $c_1, c_2\colon [0,1]^k \to M$ are two orientation preserving (i.e. $\det((c_2^{-1}c_1)') > 0$) singular k-cubes on the oriented k-dimensional manifold M and ω is a k-form on M such that $\omega = 0$ outside of $c_1([0,1]^k) \cap c_2([0,1]^k)$, then:

$$\int_{c_1} \omega = \int_{c_2} \omega\,.$$



Proof:

$$\int_{c_1} \omega = \int_{[0,1]^k} c_1^*(\omega)$$

$$= \int_{[0,1]^k} (c_2 \circ c_2^{-1} \circ c_1)^*(\omega) = \int_{[0,1]^k} (c_2^{-1} \circ c_1)^*\big(c_2^*(\omega)\big)$$

since $(f \circ g)^*\omega = g^*(f^*\omega)$.

So we need to show:

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^*\big(c_2^*(\omega)\big) = \int_{[0,1]^k} c_2^*(\omega) = \int_{c_2} \omega.$$

Let $g = c_2^{-1} \circ c_1$. If $c_2^*\omega = f dx_1 \wedge \ldots \wedge dx_k$, then:

$$\begin{aligned}(c_2^{-1} \circ c_1)^*\big(c_2^*(\omega)\big) &= g^*\big(c_2^*(\omega)\big) \\ &= g^*(f dx_1 \wedge \ldots \wedge dx_k) \\ &= (f \circ g) \det g'\, dx_1 \wedge \ldots \wedge dx_k \\ &= (f \circ g)|\det g'| dx_1 \wedge \ldots \wedge dx_k\ .\end{aligned}$$

Since $\det((c_2^{-1}c_1)') > 0$.

So we have:

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^*\big(c_2^*(\omega)\big) = \int_{[0,1]^k} (f \circ g)|\det g'| dx_1 \wedge \ldots \wedge dx_k$$

$$= \int_{c_2^{-1}c_1([0,1]^k)} c_2^*\, \omega = \int_{[0,1]^k} c_2^*\omega = \int_{c_2} \omega.$$

We still need a way to define:

$$\int_M \omega.$$

If $\omega = 0$ outside of an orientation preserving singular k-cube c in M we define:

$$\int_{\boldsymbol{M}} \boldsymbol{\omega} = \int_{\boldsymbol{c}} \boldsymbol{\omega}.$$

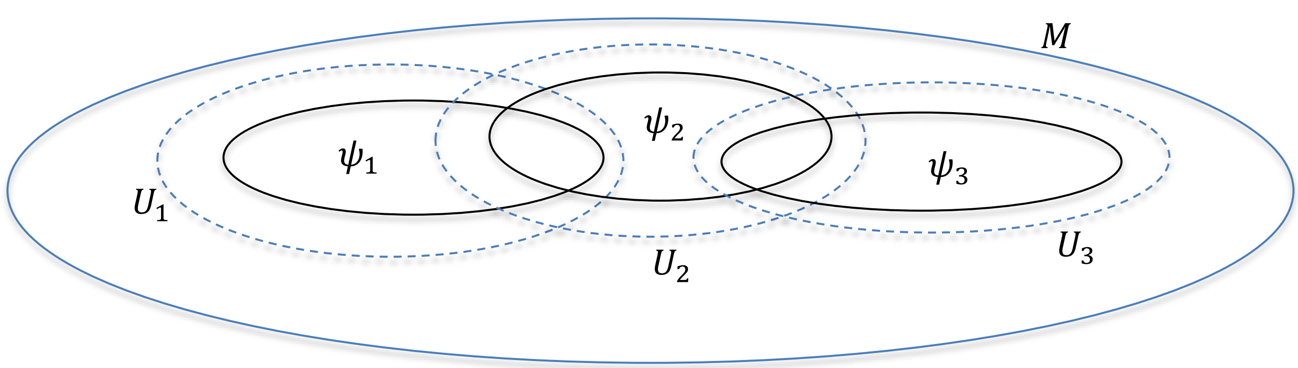
The previous theorem shows that this definition doesn't depend on which c we use. However, this still doesn't give us a general definition for:

$$\int_M \omega.$$

To define this we need the notion of a partition of unity.

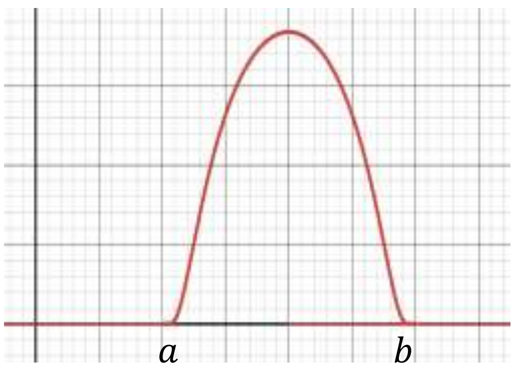
Def. Let M be a manifold and $\{U_\alpha\}_{\alpha \in I} = \mathcal{U}$ be a collection of open sets that covers M. A **partition of unity subordinate to $\boldsymbol{\mathcal{U}}$** is a collection of continuous functions $\{\psi_\alpha : M \to \mathbb{R}\}$ satisfying:

1) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in I$ and $x \in M$
2) $\psi_\alpha(x) = 0$ outside a compact subset of U_α
3) For all $x \in M$ there exists only a finite number of $\alpha \in I$ such that $\psi_\alpha(x) \neq 0$
4) $\sum_{\alpha \in I} \psi_\alpha(x) = 1$ for all $x \in M$.

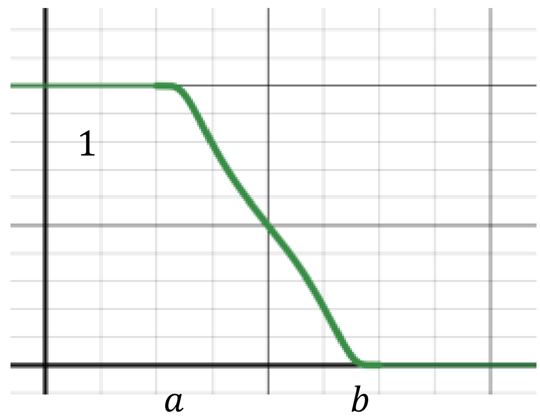


Theorem: Let M be a smooth manifold with atlas $A = \{U_\alpha, h_\alpha\}_{\alpha \in I}$.
There exists a smooth partition of unity of M subordinate to A.

A construction of a smooth partition of unity often depends on constructing smooth "bump" functions (smooth functions that are zero outside a compact set) and "cut-off" functions (smooth functions that are constant outside of a compact set).



Bump Function

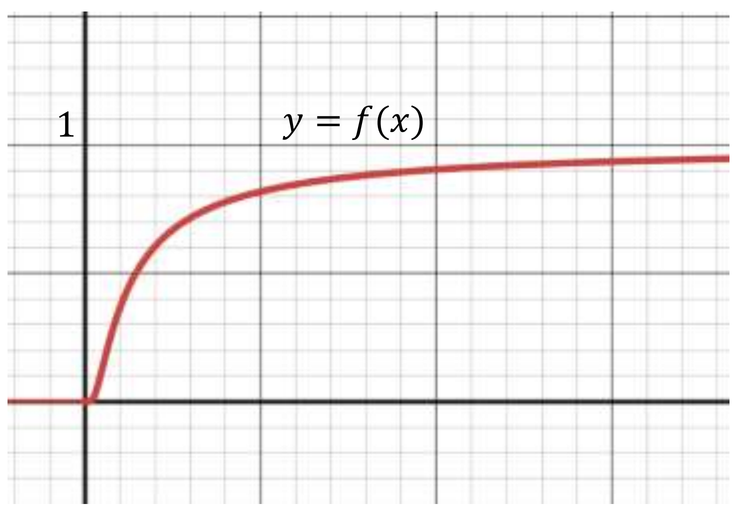


Cutoff Function

These functions can be built out of:

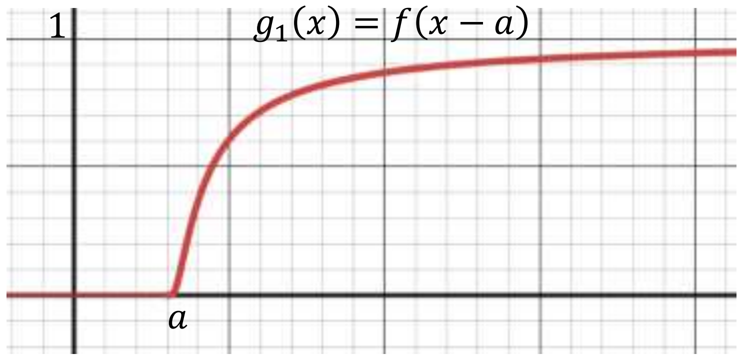
$f(x) = 0$ if $x \leq 0$

$= e^{-(\frac{1}{x})}$ if $x > 0$.

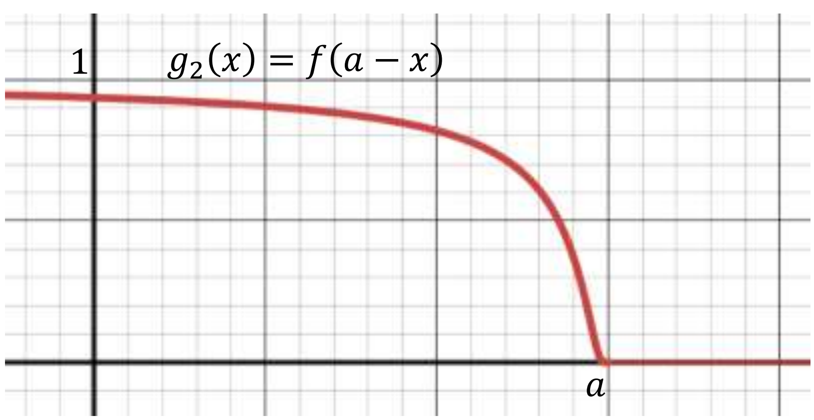


Notice:

$g_1(x) = f(x-a)$ is smooth and $g_1(x) = 0, \;\; x \le a$, and nonzero for $x > a$.



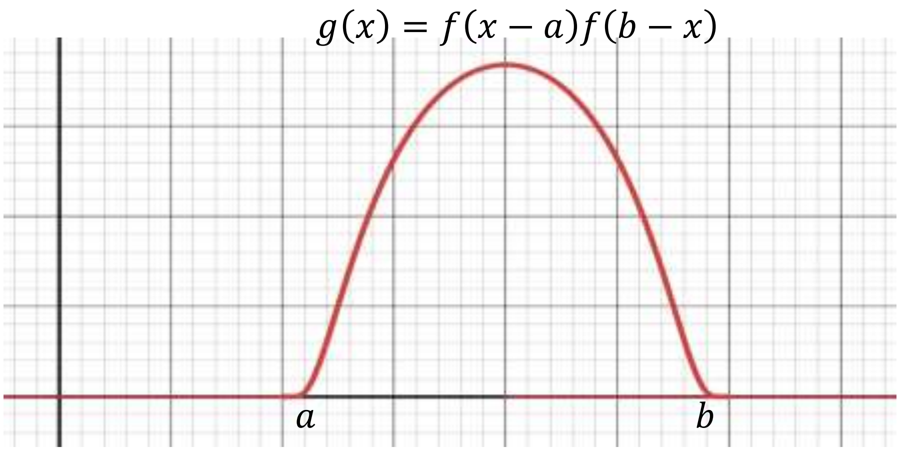
$g_2(x) = f(a-x)$ is a smooth and $g_2(x) = 0, x \ge a$ and nonzero for $x < a$.



If $a < b$, then

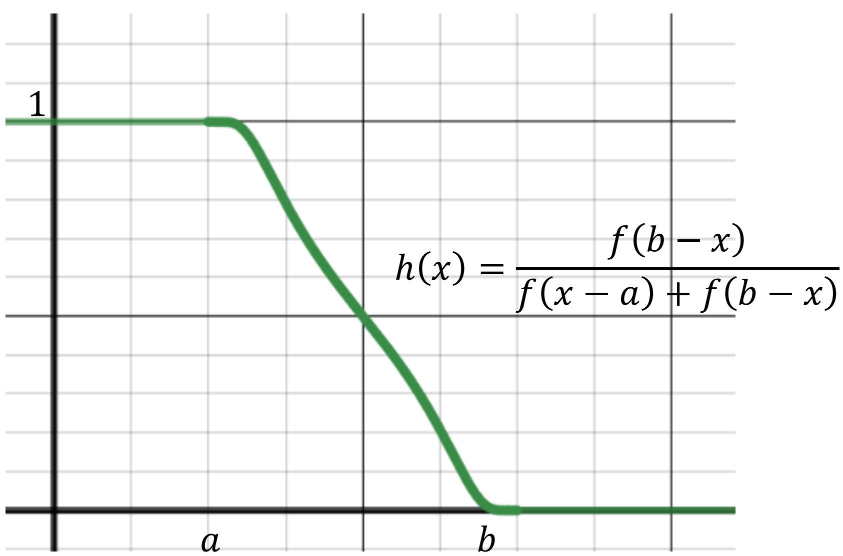
$$g(x) = f(x-a)f(b-x) = 0 \quad \text{for } x \notin (a,b)$$
$$> 0 \quad \text{for } x \in (a,b).$$

$g(x)$ is a “bump” function.



$$h(x) = \frac{f(b-x)}{f(x-a)+f(b-x)} = 1 \quad \text{if } x \le a$$
$$= 0 \quad \text{if } x \ge b.$$

$h(x)$ is strictly decreasing on (a,b) and a “cut-off” function.



Ex. Let $M = \mathbb{R}$ and let $\mathcal{U} = \{U_j\}_{j\in\mathbb{Z}}$, where $U_j = (j-1, j+1)$ be an open cover of $\mathbb{R}$. If $x \in \mathbb{Z}$, then x is only contained in U_x. If x is not an integer, then $x \in U_{[x]}$ and $x \in U_{[x]+1}$.

Consider the bump function:

$$\begin{aligned} g_j(x) &= f(x-(j-0.9))f((j+0.9)-x) \\ &= 0 \qquad \text{if } \; x \le j-0.9 \\ &= e^{\frac{1.8}{(x-j+0.9)(x-j-0.9)}} \qquad \text{if } \; j-0.9 < x < j+0.9 \\ &= 0 \qquad \text{if } \; x \ge j+0.9 \end{aligned}$$

where $f(x) = 0$ if $x \le 0$

$= e^{-\frac{1}{x}}$ if $x > 0$.

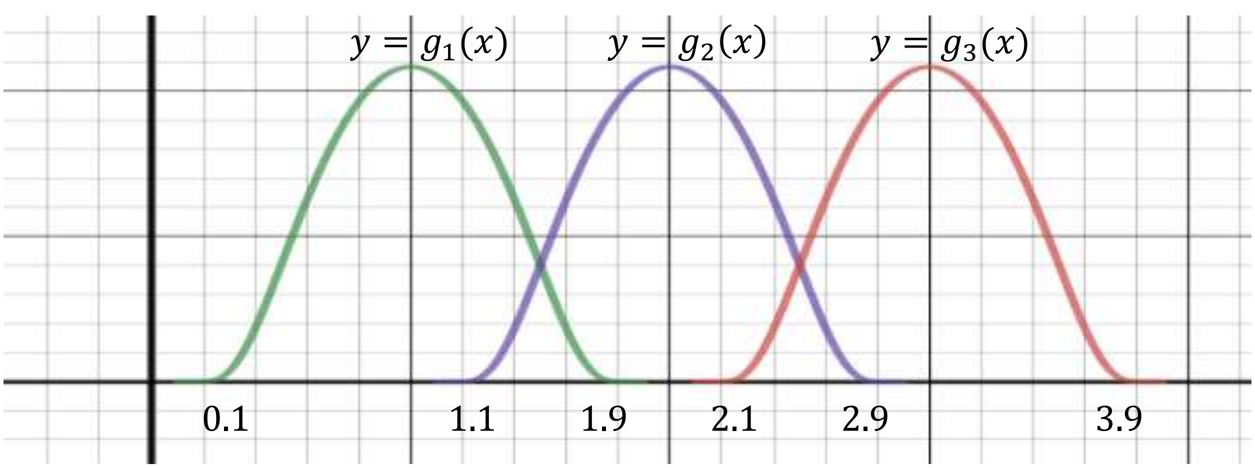
So $g_j(x) = 0$ if $x \notin [j-0.9, j+0.9]$, a compact subset of U_j. For any $j \in \mathbb{Z}$ the only functions not identically 0 on U_j are g_{j-1}, g_j, and g_{j+1}.

For example, if $j = 2, U_2 = (1,3)$, then:

$g_1(x) = 0$ if $x \notin [.1, 1.9]$

$g_2(x) = 0$ if $x \notin [1.1, 2.9]$

$g_3(x) = 0$ if $x \notin [2.1, 3.9]$



Define:

$$\psi_j(x) = \frac{g_j(x)}{g_{j-1}(x)+g_j(x)+g_{j+1}(x)}$$

$\{\psi_j(x)\}$ is a partition of unity subordinate to $\mathcal{U}$.

If $\kappa_j = [j - 0.9, j + 0.9]$, then $\psi_j(x) = 0$ if $x \notin \kappa_j$

> 0 if $x \in \kappa_j$

and $\psi_j, \psi_{j-1}, \psi_{j+1}$ are only functions not identically 0 on U_j.

If $x = n \in \mathbb{Z}$, then:

$$\sum_{j \in \mathbb{Z}} \psi_j(x) = \psi_n(n) = \frac{g_n(n)}{g_{n-1}(n) + g_n(n) + g_{n+1}(n)} = \frac{g_n(n)}{g_n(n)} = 1.$$

If $x \notin \mathbb{Z}$, then let $n = [x]$:

$$\sum_{j \in \mathbb{Z}} \psi_j(x) = \psi_n(x) + \psi_{n+1}(x)$$

$$= \frac{g_n(x)}{g_{n-1}(x)+g_n(x)+g_{n+1}(x)} + \frac{g_{n+1}(x)}{g_n(x)+g_{n+1}(x)+g_{n+2}(x)}$$

$$= \frac{g_n(x)}{g_n(x)+g_{n+1}(x)} + \frac{g_{n+1}(x)}{g_n(x)+g_{n+1}(x)} = 1$$

since $g_{n-1}(x) = g_{n+2}(x) = 0$ for $x \in U_n \cap U_{n+1}$.

Now we're ready to define $\int_M \omega$ for a general k-form ω.

Let $\mathcal{U}$ be an open cover of M such that for each $U_\alpha \in \mathcal{U}$ there is an orientation preserving singular k-cube c with $U_\alpha \subseteq c([0,1]^k)$. Now let $\{\psi_\alpha\}_{\alpha \in I}$ be a partition of unity subordinate to $\mathcal{U}$. We define:

$$\int_{\boldsymbol{M}} \boldsymbol{\omega} = \sum_{\boldsymbol{\alpha} \in \mathbf{I}} \int_{\boldsymbol{M}} (\boldsymbol{\psi_\alpha})(\boldsymbol{\omega}).$$