Differentiation and Directional Derivatives

If $f: \mathbb{R} \to \mathbb{R}$, we say that f is differentiable at $a \in \mathbb{R}$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

This definition doesn't make any sense for a function $f: \mathbb{R}^n \to \mathbb{R}^m$ (in that case, $h \in \mathbb{R}^n$ and dividing by a vector is not defined).

However, we can think of any number, $f'(a)$, as defining a linear transformation λ of $\mathbb{R}$ into $\mathbb{R}$ by:

$$\lambda: \mathbb{R} \to \mathbb{R}$$
$$\lambda(h) = \big(f'(a)\big)h.$$

So we could reformulate our definition of the derivative, $f'(a)$, by saying:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$$

Or

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \lambda(h)}{h} = 0$$

Thus, we say a function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if there is a linear transformation $\lambda: \mathbb{R} \to \mathbb{R}$ such that:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \lambda(h)}{h} = 0$$

Note: Any linear transformation, λ, of $\mathbb{R}$ into $\mathbb{R}$, $\lambda: \mathbb{R} \to \mathbb{R}$, is just multiplication by a fixed number; $\lambda(h) = ph;\ p \in \mathbb{R}$.

Now we can generalize this definition to: $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

Def. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $a \in \mathbb{R}^n$ if there is a linear transformation, $\lambda\colon \mathbb{R}^n \to \mathbb{R}^m$, such that:

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Notice that $\big(f(a+h) - f(a) - \lambda(h)\big) \in \mathbb{R}^m$ and $h \in \mathbb{R}^n$.

If this limit is 0, then we say: $Df(a) = \lambda$.

Theorem: If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then there is a unique linear transformation, $\lambda\colon \mathbb{R}^n \to \mathbb{R}^m$, such that:

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Proof: Suppose $\tau\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation that also satisfies

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \tau(h)|}{|h|} = 0.$$

Then we have:

$$0 \le \lim_{h \to 0} \frac{|\lambda(h) - \tau(h)|}{|h|}$$

$$= \lim_{h \to 0} \frac{\big|\big(\lambda(h) - f(a+h) + f(a)\big) + \big(f(a+h) - f(a) - \tau(h)\big)\big|}{|h|}$$

$$\le \lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} + \lim_{h \to 0} \frac{|f(a+h) - f(a) - \tau(h)|}{|h|}$$

$$= 0 + 0 = 0. \qquad \Rightarrow \lambda(h) = \tau(h).$$

Ex. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (xy, x + 2y)$. Using the definition of the derivative, show that:

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}.$$

We must show that $\lim_{h\to 0} \frac{\left|f\left(\vec{0}+h\right)-f\left(\vec{0}\right)-\lambda(h)\right|}{|h|} = 0$, where $\lambda = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$.

If we let $h = (h_1, h_2)$ then:

$$\lim_{h\to 0} \frac{\left|f\left(\vec{0}+h\right)-f\left(\vec{0}\right)-\lambda(h)\right|}{|h|} = \lim_{h\to 0} \frac{\left|(h_1 h_2, h_1 + 2h_2) - \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right|}{|h|}$$

$$= \lim_{h\to 0} \frac{|(h_1 h_2, h_1 + 2h_2) - (0, h_1 + 2h_2)|}{|h|} = \lim_{h\to 0} \frac{|h_1 h_2|}{\sqrt{h_1^2 + h_2^2}}$$

Notice that $(h_1 + h_2)^2 = h_1^2 + 2h_1 h_2 + h_2^2 \geq 0$

$$h_1^2 + h_2^2 \geq -2h_1 h_2$$

$$\frac{h_1^2 + h_2^2}{2} \geq |h_1 h_2| .$$

So:

$$0 \leq \lim_{h\to 0} \frac{|h_1 h_2|}{\sqrt{h_1^2 + h_2^2}} \leq \lim_{h\to 0} \frac{\frac{h_1^2 + h_2^2}{2}}{\sqrt{h_1^2 + h_2^2}} = \lim_{h\to 0} \frac{1}{2}\sqrt{h_1^2 + h_2^2} = 0$$

Thus $\lim_{h\to 0} \frac{\left|f\left(\vec{0}+h\right)-f\left(\vec{0}\right)-\lambda(h)\right|}{|h|} = 0$ and:

$$Df(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}.$$

Ex. Let $f: \mathbb{R}^2 \to \mathbb{R}$ by:

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \quad ; \quad (x,y) \neq (0,0)$$
$$= 0 \quad ; \quad (x,y) = (0,0).$$

Show f is not differentiable at $(0,0)$.

Let's assume f is differentiable at $(0,0)$ and derive a contradiction.

If f is differentiable at $(0,0)$, then there is a linear transformation:
$\lambda: \mathbb{R}^2 \to \mathbb{R}$ such that:

$$\lim_{h \to 0} \frac{|f(\vec{0}+h) - f(\vec{0}) - \lambda(h)|}{|h|} = 0$$

where $h = (h_1, h_2)$.

Let $\lambda = (a_{11} \ \ a_{12})$ so if $(x,y) \in \mathbb{R}^2$, then:

$$\lambda(x,y) = (a_{11} \ \ a_{12}) \begin{pmatrix} x \\ y \end{pmatrix} = a_{11}x + a_{12}y, \quad \text{and}$$

$$\lim_{h \to 0} \frac{\left| \frac{h_1 \, h_2}{\sqrt{h_1^2+h_2^2}} - (a_{11}h_1 + a_{12}h_2) \right|}{\sqrt{h_1^2+h_2^2}} = 0.$$

For this limit to exist we must get the same value, 0, no matter which direction h approaches $(0,0)$.

Suppose $h = (h_1, 0)$; i.e. we approach $(0,0)$ along the x-axis.

$$\lim_{h_1 \to 0} \frac{|-a_{11}h_1|}{\sqrt{h_1^2}} = \lim_{h_1 \to 0} \frac{|a_{11}||h_1|}{|h_1|} = |a_{11}| = 0$$

so $a_{11} = 0$.

Now approach $(0,0)$ along the y-axis, i.e. $h_1 = 0$.

$$\lim_{h_2 \to 0} \frac{|-a_{12}h_2|}{\sqrt{h_2^2}} = \lim_{h_2 \to 0} \frac{|a_{12}||h_2|}{|h_2|} = |a_{12}| = 0$$

so $a_{12} = 0$.

Thus, $\lambda = (a_{11} \quad a_{12}) = (0 \quad 0)$ maps all vectors in $\mathbb{R}^2$ to 0.

Knowing $\lambda = (0 \quad 0)$, let's approach $(0,0)$ by
$h = (h_1, h_1)$, i.e. $h_2 = h_1$.

$$\lim_{h_1 \to 0} \frac{\left|\frac{h_1^2}{\sqrt{h_1^2 + h_1^2}} - 0\right|}{\sqrt{h_1^2 + h_1^2}} = \lim_{h_1 \to 0} \frac{h_1^2}{(h_1^2 + h_2^2)} = \lim_{h_1 \to 0} \frac{h_1^2}{2h_1^2} = \frac{1}{2}.$$

But then:

$$\lim_{h \to 0} \frac{\left|f(\vec{0} + h) - f(\vec{0}) - \lambda(h)\right|}{|h|} \neq 0$$

thus, $f(x, y)$ does not have a derivative at $(0,0)$.

Ex. Let $f: \mathbb{R}^2 \to \mathbb{R}$ by:

$$f(x,y) = \frac{x^2y^4}{x^4+6y^8} \qquad (x,y) \neq (0,0)$$
$$= 0 \qquad (x,y) = (0,0)$$

Show that $f(x,y)$ is not differentiable at $(0,0)$.

Let's assume $f(x,y)$ is differentiable at $(0,0)$ and derive a contradiction.

If f is differentiable at $(0,0)$, then there is a linear transformation, $\lambda: \mathbb{R}^2 \to \mathbb{R}$, where:

$$\lambda(x,y) = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = a_{11}x + a_{12}y$$

and

$$\lim_{h \to 0} \frac{|f(\vec{0}+h) - f(\vec{0}) - \lambda(h)|}{|h|} = 0.$$

If we let $h = (h_1, h_2)$ then:

$$\lim_{h \to 0} \frac{\left| \dfrac{h_1^2 h_2^4}{h_1^4 + 6h_2^8} - (a_{11}h_1 + a_{12}h_2) \right|}{\sqrt{h_1^2 + h_2^2}} = 0.$$

For this limit to exist, we must get the same value, 0, when approaching $(0,0)$ from any direction. In particular, suppose $h = (h_1, 0)$ (i.e. we approach $(0,0)$ along the x-axis).

$$\lim_{h_1 \to 0} \frac{|-a_{11}h_1|}{\sqrt{h_1^2}} = 0 \quad \Rightarrow \quad a_{11} = 0.$$

Now approach $(0,0)$ along the y-axis (i.e. $h_1 = 0$):

$$\lim_{h_2 \to 0} \frac{|-a_{12}h_2|}{\sqrt{h_2^2}} = 0 \quad \Rightarrow \quad a_{12} = 0$$

Thus, $\lambda = (a_{11} \quad a_{12}) = (0 \quad 0)$.

Knowing $\lambda = (0 \quad 0)$, let's approach $(0,0)$ by $h_1 = h_2^2$ (i.e. $h = (h_2^2, h_2)$).

$$\lim_{h \to 0} \frac{\left|f\left(\vec{0}+h\right) - f\left(\vec{0}\right) - \lambda(h)\right|}{|h|} = \lim_{h_2 \to 0} \frac{h_2^4 h_2^4}{(h_2^8 + 6h_2^8)\sqrt{h_2^4 + h_2^2}}$$

$$= \lim_{h_2 \to 0} \frac{h_2^8}{(7h_2^8)\sqrt{h_2^4 + h_2^2}}$$

$$= \lim_{h_2 \to 0} \frac{1}{(7)\sqrt{h_2^4 + h_2^2}} \neq 0.$$

Thus:

$$\lim_{h \to 0} \frac{\left|f\left(\vec{0}+h\right) - f\left(\vec{0}\right) - \lambda(h)\right|}{|h|} \neq 0$$

and $Df(0,0)$ does not exist.

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then it's continuous at $a \in \mathbb{R}^n$.

Proof: To show f is continuous at $a \in \mathbb{R}^n$ we need to show:

$\lim_{x \to a} f(x) = f(a)$ or equivalently $\lim_{h \to 0}\big(f(a+h) - f(a)\big) = 0.$

We need to use the fact that $Df(a)$ exists:

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

for some linear transformation $\lambda: \mathbb{R}^n \to \mathbb{R}^m$.

Notice that:

$$\begin{aligned} 0 \leq |f(a+h) - f(a)| &= |f(a+h) - f(a) - \lambda(h) + \lambda(h)| \\ &\leq |f(a+h) - f(a) - \lambda(h)| + |\lambda(h)| \\ &= \tfrac{|f(a+h) - f(a) - \lambda(h)|}{|h|} \cdot |h| + |\lambda(h)|. \end{aligned}$$

For any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, we know there is a $M \in \mathbb{R}$ such that:

$$|T(h)| \leq M|(h)|.$$

Thus:

$$0 \leq |f(a+h) - f(a)| \leq \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} \, |h| + M|h|.$$

Letting $h \to 0$ we know the RHS becomes 0 (why?).
Thus by the squeeze theorem:

$$\lim_{h \to 0} |f(a+h) - f(a)| = 0.$$

Hence:

$$\lim_{h \to 0} (f(a+h) - f(a)) = 0$$

$$\lim_{h \to 0} f(a+h) = f(a).$$

Differentiation Theorems:

The Chain Rule: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, and $g: \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(a)$, then $g \circ f: \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a, and:

$$D(g \circ f)(a) = Dg\big(f(a)\big) \circ Df(a).$$

1) If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a constant function, then

   $$Df(a) = 0$$

2) If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

   $$Df(a) = f$$

3) If $f: \mathbb{R}^n \to \mathbb{R}^m$, then f is differentiable at $a \in \mathbb{R}^n$ if, and only if, f_i is differentiable and $Df(a) = \big(Df_1(a), \dots , Df_m(a)\big)$. Thus, $Df(a)$ is the $m \times n$ matrix whose i^{th} row is $Df_i(a)$

4) If $g: \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x, y) = x + y$, then

   $$Dg(a, b) = g$$

5) If $m: \mathbb{R}^2 \to \mathbb{R}$ is defined by $m(x, y) = xy$, then

   $\big(Dm(a, b)\big)(x, y) = bx + ay$, thus $Dm(a, b) = (b \quad a)$.

Proof:

1.

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-0|}{|h|}=0$$

2.

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-f(h)|}{|h|}=\lim_{h\to 0}\frac{|f(a)+f(h)-f(a)-f(h)|}{|h|}=0$$

3. If f_i is differentiable at a and $\lambda = \big(Df_1(a),\dots,Df_m(a)\big)$, then:

$$\begin{aligned}&f(a+h)-f(a)-\lambda(h)\\&\quad=\big(f_1(a+h)-f_1(a)-Df_1(a)(h),\dots,f_m(a+h)-f_m(a)-Df_m(a)(h)\big)\end{aligned}$$

So,

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-\lambda(h)|}{|h|}\le\lim_{h\to 0}\sum_{i=1}^{m}\frac{|f_i(a+h)-f_i(a)-Df_i(a)(h)|}{|h|}=0.$$

Thus f is differentiable at a and $Df(a)=\lambda$.

If f is differentiable at a, then by #2 and the chain rule, $f_i=\pi_i\circ f$ is differentiable at a where $\pi_i(x)=x_i$, and

$$\begin{aligned}Df_i(a)&=D\pi_i(f(a))\circ Df(a)\\&=\pi_i\circ Df(a)\end{aligned}$$

Thus $Df(a)=\big(Df_1(a),\dots,Df_m(a)\big)$.

4. Since $g(x,y)=x+y$ is a linear transformation from $\mathbb{R}^2\to\mathbb{R}$, it follows from #2 that $Dg(a,b)=g$.

5. Let $\lambda(x,y) = bx + ay$, then

$$\lim_{h\to 0}\frac{|m(a+h_1,b+h_2)-m(a,b)-\lambda(h_1,h_2)|}{|(h_1,h_2)|} = \lim_{h\to 0}\frac{|h_1\ h_2|}{\sqrt{h_1^2+h_2^2}}.$$

Notice that:

$$|h_1\ h_2| \le |h_1\ |^2 \quad \text{if } |h_2| \le |h_1|$$

$$|h_1\ h_2| \le |h_2\ |^2 \quad \text{if } |h_1| \le |h_2|$$

Hence: $$|h_1\ h_2| \le |h_1|^2 + |h_2|^2.$$

So we can write:

$$0 \le \frac{|h_1\ h_2|}{\sqrt{h_1^2+h_2^2}} \le \frac{h_1^2+h_2^2}{\sqrt{h_1^2+h_2^2}} = \sqrt{h_1^2+h_2^2}$$

$$\Rightarrow \qquad \lim_{h\to 0}\frac{|h_1\ h_2|}{\sqrt{h_1^2+h_2^2}} = 0.$$

Corollary: If $f, g: \mathbb{R}^n \to \mathbb{R}$ are differentiable at a, then

i) $D(f+g)(a) = Df(a) + Dg(a)$
ii) $D(f\cdot g)(a) = g(a)Df(a) + f(a)Dg(a)$
iii) If $g(a) \neq 0$, then:

$$D\left({}^{f}/_{g}\right)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{\left(g(a)\right)^2}$$

Proof of ii:

Let $F: \mathbb{R}^n \to \mathbb{R}^2$ by $F(x) = \big(f(x), g(x)\big)$
$p: \mathbb{R}^2 \to \mathbb{R}$ by $p(x_1, x_2) = x_1 \cdot x_2$
then, $f(x)g(x) = p \circ F(x)$.

$$\begin{aligned} D(fg)(a) &= D(p \circ F)(a) \\ &= Dp\big(F(a)\big) \circ DF(a) && \text{Chain Rule} \\ &= Dp\big(f(a), g(a)\big) \circ DF(a) \\ &= \big(g(a) \;\; f(a)\big)\begin{pmatrix} Df(a) \\ Dg(a) \end{pmatrix} && \text{by \#3, \#5} \\ &= g(a)Df(a) + f(a)Dg(a). \end{aligned}$$

In second year calculus, given a function, $f: \mathbb{R}^3 \to \mathbb{R}$, we define the directional derivative of f at $x \in \mathbb{R}^3$ along a unit vector, $\vec{u}$, as:

$$D_{\vec{u}}f(x) = \frac{d}{dt} f(x + t\vec{u})\Big|_{t=0}.$$

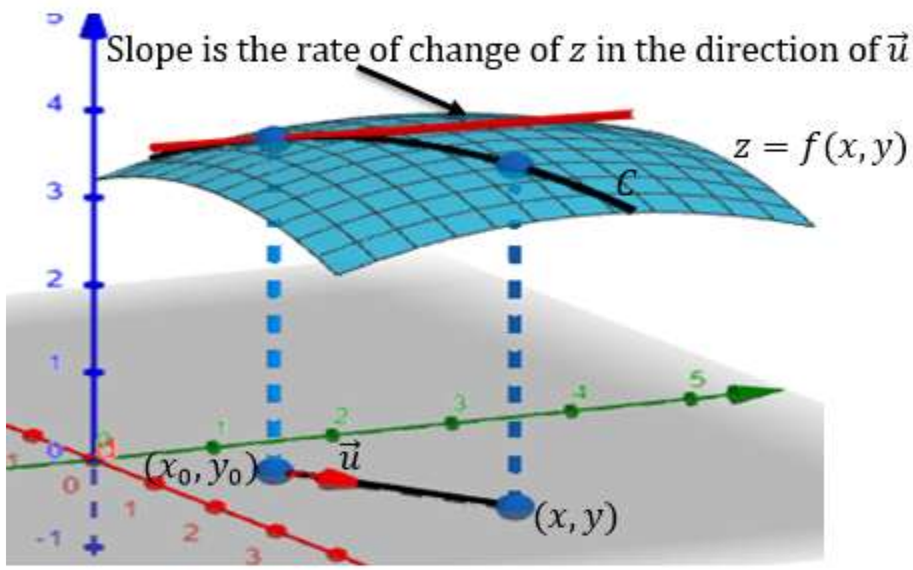
Furthermore, you learn that it can be calculated by $D_{\vec{u}}f(x) = \nabla \mathrm{f} \cdot \vec{u}$.
This follows from the chain rule. If we let $\vec{u} = (u_1, u_2, u_3)$:

$$\begin{aligned} \frac{d}{dt} f(x + t\vec{u}) &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\frac{dx_3}{dt} \\ &= \left(\frac{\partial f}{\partial x_1}\right)(u_1) + \left(\frac{\partial f}{\partial x_2}\right)(u_2) + \left(\frac{\partial f}{\partial x_3}\right)(u_3) \\ &= \nabla \mathrm{f} \cdot \vec{u}. \end{aligned}$$

This statement is true for all t, hence it's true for $t = 0$. Thus

$$D_{\vec{u}}f(x) = \nabla \mathrm{f} \cdot \vec{u}.$$

The Directional Derivative represents the rate of change in the value of $z = f(x, y)$ in the direction of $\vec{u}$.



We can generalize this notion to define a directional derivative for a function $F: \mathbb{R}^n \to \mathbb{R}^m$.

Def. Let $F: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $x \in U$, and $\vec{u}$ be a unit vector in $\mathbb{R}^n$. The **directional derivative** of F in the direction of $\vec{u}$ at $x_0 \in U$ is:

$$D_{\vec{u}}F(x) = \lim_{h \to 0} \frac{F(x+h\vec{u}) - F(x)}{h}.$$

Notice that $D_{\vec{u}}F(x)$ is a vector in $\mathbb{R}^m$, whereas the directional derivative of a function, $f: \mathbb{R}^3 \to \mathbb{R}$, at a point was a number (or a vector with one component).

However, it is still the case that:

$$D_{\vec{u}}F(x) = \frac{d}{dt}\big(F(x + t\vec{u})\big)\Big|_{t=0}.$$

We can see this by letting $g(t) = F(x + t\vec{u})$:

$$g'(0) = \lim_{h\to 0}\frac{g(0+h) - g(0)}{h} = \lim_{h\to 0}\frac{F(x + h\vec{u}) - F(x)}{h} = D_{\vec{u}}F(x).$$

Thus: $$D_{\vec{u}}F(x) = \tfrac{d}{dt}\big(F(x + t\vec{u})\big)\Big|_{t=0}.$$

Ex. Let $F(x,y) = (x^2 - y^2, 2xy)$. Find the directional derivative of F at $(1,2)$ in the direction of $\vec{u} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

$$F(x + t\vec{u}) = F\left((1,2) + t\left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right)\right) = F\left(1 + \tfrac{t}{2}, 2 - \tfrac{\sqrt{3}}{2}t\right)$$

$$= \left(\left(1 + \tfrac{t}{2}\right)^2 - \left(2 - \tfrac{\sqrt{3}}{2}t\right)^2, 2\left(1 + \tfrac{t}{2}\right)\left(2 - \tfrac{\sqrt{3}}{2}t\right)\right).$$

$$D_{\vec{u}}F(1,2) = \tfrac{d}{dt}\left(F((1,2) + t\left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right))\right)\Big|_{t=0}.$$

$$\tfrac{d}{dt}\left(F((1,2) + t\left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right))\right)$$
$$= \left(\left(2\left(1 + \tfrac{t}{2}\right)\left(\tfrac{1}{2}\right) - 2\left(2 - \tfrac{\sqrt{3}}{2}t\right)\left(-\tfrac{\sqrt{3}}{2}\right)\right), 2\left(1 + \tfrac{t}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right) + 2\left(\tfrac{1}{2}\right)\left(2 - \tfrac{\sqrt{3}}{2}t\right)\right).$$

So at $t = 0$: $\frac{dF}{dt}\Big|_{t=0} = \left(1 + 2\sqrt{3},\ -\sqrt{3} + 2\right) = D_{\vec{u}}\big(F(1,2)\big)$.

There is also a similar formula to the case where $f: \mathbb{R}^3 \to \mathbb{R}$ (i.e. $D_{\vec{u}} f(x) = \nabla \mathrm{f} \cdot \vec{u}$) where $F: \mathbb{R}^n \to \mathbb{R}^m$.

Notice that

$$\frac{d}{dt}\big(F(x+t\vec{u})\big) = \frac{d}{dt}\left(F_1(x+t\vec{u}), F_2(x+t\vec{u}), \dots, F_m(x+t\vec{u})\ \right)$$

where $F_i: \mathbb{R}^n \to \mathbb{R}$.

Thus we have:

$$\frac{d}{dt}\big(F(x+t\vec{u})\big) = (\nabla \mathrm{F}_1 \cdot \vec{u},\ \nabla \mathrm{F}_2 \cdot \vec{u}, \dots,\ \nabla \mathrm{F}_\mathrm{m} \cdot \vec{u}).$$

This is again true for all t so it is true for $t = 0$ and

$$D_{\vec{u}} f(x) = (\nabla \mathrm{F}_1 \cdot \vec{u}, \nabla \mathrm{F}_2 \cdot \vec{u}, \dots, \nabla \mathrm{F}_\mathrm{m} \cdot \vec{u})\ .$$

We will see soon that:

$$DF(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

Thus we have:

$$(DF(x))(\vec{u}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$= (\nabla \mathrm{F}_1 \cdot \vec{u}, \nabla \mathrm{F}_2 \cdot \vec{u}, \dots, \nabla \mathrm{F}_\mathrm{m} \cdot \vec{u})$$

$$= D_{\vec{u}} F(x).$$

Thus:

$$D_{\vec{u}} F(x) = \big(DF(x)\big)(\vec{u}).$$

Ex. Let $F(x,y) = (x^2 - y^2, 2xy)$, $(x,y) = (1,2)$, and $\vec{u} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.
Find $D_{\vec{u}}F(1,2)$.

$$DF(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(x,y) & \frac{\partial F_1}{\partial y}(x,y) \\ \frac{\partial F_2}{\partial y}(x,y) & \frac{\partial F_2}{\partial y}(x,y) \end{pmatrix}$$

Thus,

$$DF(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

So at $(1,2)$:

$$DF(1,2) = \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}$$

$$D_{\vec{u}}F(1,2) = \big(DF(1,2)\big)\begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 + 2\sqrt{3} \\ 2 - \sqrt{3} \end{pmatrix}.$$

If the derivative of a function exists then the directional derivative exists in every direction and $D_{\vec{u}}F(x) = \big(DF(x)\big)(\vec{u})$.

However, the directional derivative might exist in every direction without the derivative existing.

Ex. Let:

$$f(x,y) = \frac{xy^2}{x^2+y^4} \quad \text{if } (x,y) \neq (0,0)$$
$$= 0 \quad \text{if } (x,y) = (0,0)$$

Show the directional derivative at $(0,0)$ exists in all directions but $Df(0,0)$ does not exist.

Let $\vec{u} = (a,b)$ be any unit vector.

$$D_{\vec{u}}f(0,0) = \lim_{h\to 0}\frac{f\left(\vec{0}+h\vec{u}\right) - f\left(\vec{0}\right)}{h} = \lim_{h\to 0}\frac{h^3ab^2}{h(h^2a^2+h^4b^4)}$$

$$= \lim_{h\to 0}\frac{ab^2}{(a^2+h^2b^4)} = \frac{ab^2}{a^2} = \frac{b^2}{a}.$$

If $a = 0$, then $f\left(\vec{0}+h\vec{u}\right) = 0$ for all h and thus $D_{\vec{u}}f(0,0) = 0$. Thus, $D_{\vec{u}}f(0,0)$ exists in all directions.

We can show that $Df(0,0)$ doesn't exist in two ways.

1. $Df(0,0)$ does not exist because $f(x,y)$ is not continuous at $(0,0)$. We can see this since approaching $(0,0)$ by letting
   $h_1 = h_2^2$, we get:

$$\lim_{h_2\to 0} f(h_2^2, h_2) = \lim_{h_2\to 0}\frac{h_2^2\, h_2^2}{h_2^4 + h_2^4} = \frac{1}{2} \neq f(0,0) = 0$$

So $f(x,y)$ is not continuous at $(0,0)$, hence $Df(0,0)$ does not exist.

2. The second way to see this is to assume that $Df(0,0)$ exists and get a contradiction. If $Df(0,0)$ exists, then there is a linear transformation $\lambda: \mathbb{R}^2 \to \mathbb{R}, \lambda(x,y) = a_{11}x + a_{12}y$ such that:

$$0 = \lim_{h\to 0} \frac{\left|f\left(\vec{0}+h\right) - f\left(\vec{0}\right) - \lambda(h)\right|}{|h|} = \lim_{h\to 0} \frac{\left|\frac{h_1 h_2^2}{h_1^2 + h_2^4} - (a_{11}h_1 + a_{12}h_2)\right|}{|h|}.$$

If $h \to 0$ along the x-axis, i.e. $h_2 = 0$, then the limits is:

$$0 = \lim_{h_1\to 0} \frac{|-a_{11}h_1|}{|h_1|} \quad \Rightarrow \quad a_{11} = 0$$

If $h \to 0$ along the y-axis, i.e. $h_1 = 0$, then the limit is:

$$0 = \lim_{h_2\to 0} \frac{|-a_{12}h_2|}{|h_2|} \quad \Rightarrow \quad a_{12} = 0$$

But this implies $\lambda = (\,0 \quad 0\,)$.

Now if $h \to 0$ along (h_2^2, h_2), then:

$$0 = \lim_{h_2\to 0} \frac{\left|\frac{h_2^2\, h_2^2}{h_2^4 + h_2^4}\right|}{\sqrt{h_2^4 + h_2^2}} = \lim_{h_2\to 0} \frac{\frac{1}{2}}{\sqrt{h_2^4 + h_2^2}} \neq 0$$

which is a contradiction, so $Df(0,0)$ does not exist.