Vector Fields and Differential Forms on Manifolds

Def. Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold. A **vector field on $\boldsymbol{M}$** is a function on M that associates to each point $p \in M$ a vector $F(p) \in T_pM$.

Let $\vec{\Phi}: U \to M$ be a parametrization. Given a vector field $F(x)$ on M, there is a unique vector field G on U such that:

$$\vec{\Phi}_*\big(G(a)\big) = F\left(\vec{\Phi}(a)\right)$$

for $a \in U$, and where:

$$\vec{\Phi}_*\big(G(a)\big) = \left(D\vec{\Phi}(a)\big(G(a)\big)\right)_{\vec{\Phi}(a)}.$$

We say F is differentiable if G is differentiable. Note that the definition of differentiability of F does not depend on which parametrization is used.

Ex. Let M be parametrized by $\vec{\Phi}(u,v) = (u, v, u^2 + v^2)$. Then at each point $p = (u, v, u^2 + v^2)$ on M, the tangent space T_pM has a basis of $\left\{\frac{\partial\vec{\Phi}}{\partial u}, \frac{\partial\vec{\Phi}}{\partial v}\right\} = \{(1,0,2u), (0,1,2v)\}$. Thus we can express any vector field on M as:

$$F(p) = f_1(p)\frac{\partial\vec{\Phi}}{\partial u} + f_2(p)\frac{\partial\vec{\Phi}}{\partial v};$$

where f_1 and f_2 are real valued function on M.

For example, $F(u,v) = uv\frac{\partial\vec{\Phi}}{\partial u} + (u - v)\frac{\partial\vec{\Phi}}{\partial v}$
Is a vector field on M.

Def. A function ω, which assigns $\omega(x) \in \Omega^p(T_xM)$ for each $x \in M$, is called a **$\boldsymbol{p}$-form on $\boldsymbol{M}$**.

If $\overrightarrow{\Phi}: U \to M$ is a parametrization, then $\overrightarrow{\Phi}^*(\omega)$ is a p-form on U. We say $\boldsymbol{\omega}$ **is differentiable** if $\overrightarrow{\Phi}^*(\omega)$ is differentiable.

A p-form on M can be written as:

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1,\ldots,i_p}\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

where the functions $\omega_{i_1,\ldots,i_p}$ are defined on M.

Theorem: There is a unique $(p+1)$-form $d\omega$ on M such that for every parametrization $\overrightarrow{\Phi}: W \to M \subseteq \mathbb{R}^n$ we have:

$$\left(\overrightarrow{\Phi}\right)^*(d\omega) = d\left(\overrightarrow{\Phi}^*(\omega)\right).$$

Proof: If $\overrightarrow{\Phi}: W \subseteq \mathbb{R}^k \to \mathbb{R}^n$ is a parametrization with $x = \overrightarrow{\Phi}(a)$ and $v_1, \ldots, v_{p+1} \in T_x M$, then there are unique $w_1, \ldots, w_{p+1} \in \mathbb{R}^k_a$ such that:

$$\overrightarrow{\Phi}_*(w_i) = D\overrightarrow{\Phi}(a)(w_i) = v_i$$

since $D\overrightarrow{\Phi}(a)$ is an invertible linear map from $\mathbb{R}^k_a$ onto $T_x M$.

We define:

$$\boldsymbol{d\omega(x)\left(v_1, \ldots, v_{p+1}\right) = d\left(\overrightarrow{\Phi}^*\omega\right)(a)\left(w_1, \ldots, w_{p+1}\right).}$$

One can check that this doesn't depend on the parametrization chosen, so $d\omega$ is well defined.

Ex. Let M be parametrized by $\overrightarrow{\Phi}(u,v) = (u, v, u^2 + v^2)$.
Then a 1-form on M has the form:

$$\omega = f_1(u,v)du + f_2(u,v)dv$$

where

$$du\left(\frac{\partial\overrightarrow{\Phi}}{\partial u}\right) = 1, \quad du\left(\frac{\partial\overrightarrow{\Phi}}{\partial v}\right) = 0$$
$$dv\left(\frac{\partial\overrightarrow{\Phi}}{\partial u}\right) = 0, \quad dv\left(\frac{\partial\overrightarrow{\Phi}}{\partial v}\right) = 1.$$

For example:

$$\omega = (uv)du + (u - v)dv$$

Is a 1-form on M.

A 2-form on M has the form:

$$\eta = f(u,v)dudv.$$

For example:

$$\eta = (u^2 - uv)dudv$$

Is a 2-form on M.