Representing Tangent Spaces of Manifolds

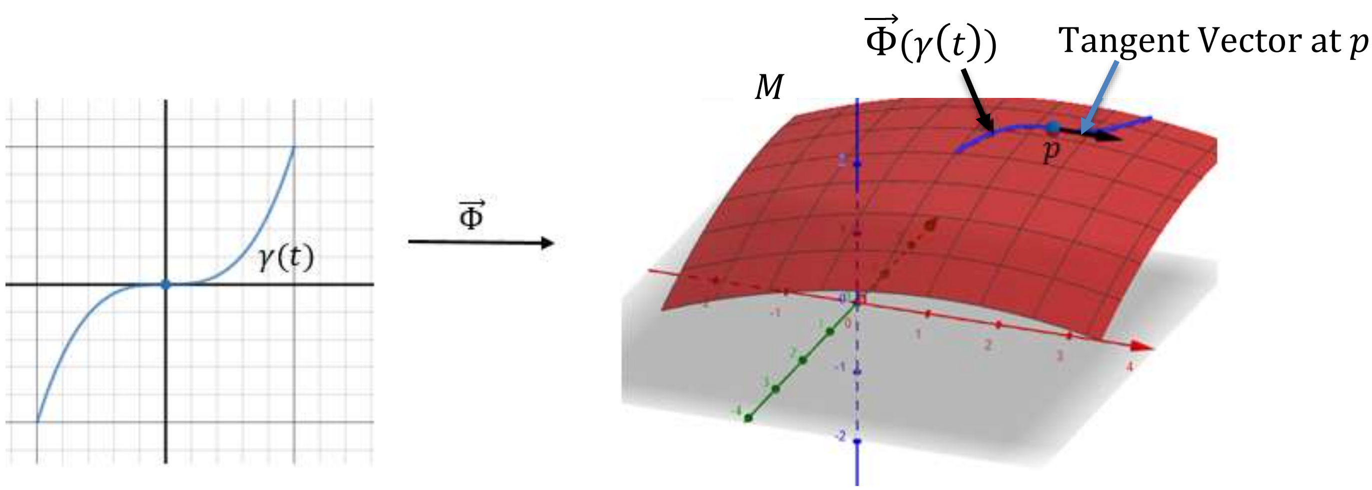
Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold and $\overrightarrow{\Phi}$ a parametrization where $\overrightarrow{\Phi}: U \subseteq \mathbb{R}^k \to M \subseteq \mathbb{R}^n$ and $\overrightarrow{\Phi}(a) = x \in M$. We know that:

$$D\overrightarrow{\Phi}(a): \mathbb{R}^k_a \to \mathbb{R}^n_x \; .$$

We call $D\overrightarrow{\Phi}(\mathbb{R}^k_a) = T_x(M)$ the **tangent space of M at x**. In fact this definition does not depends on the parametrization of $\overrightarrow{\Phi}$.

Def. A **tangent vector** to a manifold, M, at a point, $p \in M$, is the tangent vector at p of a curve in M passing through p.

The **tangent space** of M at p, T_pM, is also the set of all tangent vectors to M at p.



Ex. Find an equation of the tangent space (i.e. plane) to the unit sphere $x^2 + y^2 + z^2 = 1$ at the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

We first need to find a parametrization of the unit sphere, it doesn't matter which one we use, and the point that gets mapped to $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

Let's use the parametrization $\overrightarrow{\Phi}(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$.
Clearly $u = \frac{1}{2}$, $v = \frac{1}{2}$ gets mapped to $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

By the definition of the tangent space, it is the image of $\mathbb{R}^2_{\left(\frac{1}{2}, \frac{1}{2}\right)}$ under $D\overrightarrow{\Phi}\left(\frac{1}{2}, \frac{1}{2}\right)$. So we just need to find the image under $D\overrightarrow{\Phi}\left(\frac{1}{2}, \frac{1}{2}\right)$ of the basis vectors $(1, 0)$ and $(0, 1)$ for $\mathbb{R}^2_{\left(\frac{1}{2}, \frac{1}{2}\right)}$.

$$D\overrightarrow{\Phi}(u, v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} \overrightarrow{\Phi}_u & \overrightarrow{\Phi}_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-u}{\sqrt{1-u^2-v^2}} & \frac{-v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$

$$D\overrightarrow{\Phi}\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{pmatrix}.$$

So a basis for $T_{\left(\frac{1}{2},\frac{1}{2},\frac{-\sqrt{2}}{2}\right)}S^2$ is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{-\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{-\sqrt{2}}{2} \end{pmatrix}.$$

So the vectors $\vec{w}_1 = \left(1, 0, -\frac{\sqrt{2}}{2}\right)$ and $\vec{w}_2 = \left(0, 1, -\frac{\sqrt{2}}{2}\right)$ span the tangent space to S^2 at $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

To find an equation for the tangent plane we need a normal vector $(\vec{w}_1 \times \vec{w}_2)$ and a point on the plane $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

$$\vec{w}_1 \times \vec{w}_2 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right).$$

So an equation of the tangent plane to S^2 at $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ is:

$$\frac{\sqrt{2}}{2}\left(x - \frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(y - \frac{1}{2}\right) + \left(z - \frac{\sqrt{2}}{2}\right) = 0.$$

Ex. Consider the 3 dimensional manifold in $\mathbb{R}^4$ given by:

$$M = \{(x_1, x_2, x_3, x_4) |\ x_4 = x_1^2 + x_2^2 + x_3^2\}$$

Find 3 vectors that span T_pM where $p = (1, 2, 1, 6)$. Then find an equation for the tangent space.

We can parametrize this manifold by:

$$\vec{\Phi}(u_1, u_2, u_3) = (u_1, u_2, u_3, u_1^2 + u_2^2 + u_3^2), \quad (u_1, u_2, u_3) \in \mathbb{R}^3.$$

We need to find the image of $\mathbb{R}^3_{(1,2,1)}$ under $D\vec{\Phi}(1, 2, 1)$.

$$D\vec{\Phi}(u_1, u_2, u_3) = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_2} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \\ \frac{\partial x_4}{\partial u_1} & \frac{\partial x_4}{\partial u_2} & \frac{\partial x_4}{\partial u_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2u_1 & 2u_2 & 2u_3 \end{pmatrix}$$

$$D\vec{\Phi}(1, 2, 1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & 2 \end{pmatrix}.$$

$(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ are a basis for $\mathbb{R}^3$ so:

$$D\vec{\Phi}(1, 2, 1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$D\vec{\Phi}(1,2,1)\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\2 & 4 & 2\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\4\end{pmatrix}$$

$$D\vec{\Phi}(1,2,1)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\2 & 4 & 2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\2\end{pmatrix}.$$

So $\vec{w}_1 = (1, 0\ , 0, 2)$, $\vec{w}_2 = (0, 1, 0, 4)$, $\vec{w}_3 = (0, 0, 1, 2)$ span the tangent space to M at $(1, 2, 1, 6)$.

If we can find a normal vector to these 3 vectors in $\mathbb{R}^4$, then we can find an equation for the tangent space. Unfortunately, we don't have a cross product in $\mathbb{R}^4$. However, if $\vec{w} = (a, b, c, d)$ is perpendicular to the 3 vectors, then:

$$\vec{w} \cdot \vec{w}_1 = (a, b, c, d) \cdot (1, 0\ , 0, 2) = a + 2d = 0 \quad \Rightarrow d = -\frac{a}{2}$$

$$\vec{w} \cdot \vec{w}_2 = (a, b, c, d) \cdot (0, 1, 0, 4) = b + 4d = 0 \quad \Rightarrow d = -\frac{b}{4}$$

$$\vec{w} \cdot \vec{w}_3 = (a, b, c, d) \cdot (0, 0, 1, 2) = c + 2d = 0 \quad \Rightarrow d = -\frac{c}{2}$$

$$-\frac{a}{2} = -\frac{b}{4} \quad \Rightarrow \quad b = \frac{a}{2}$$

$$-\frac{a}{2} = -\frac{c}{2} \quad \Rightarrow \quad c = a$$

$$d = -\frac{a}{2}$$

We can choose any number for a, so let $a = 1$, then:

$\vec{w} = (a, b, c, d) = \left(1, \frac{1}{2}, 1, -\frac{1}{2}\right)$ is normal to the tangent space.

Since we are finding the tangent space at the point $p = (1, 2, 1, 6)$, we can represent any vector in the tangent space by $(x_1 - 1, x_2 - 2, x_3 - 1, x_4 - 6)$. Since $\vec{w} = \left(1, \frac{1}{2}, 1, -\frac{1}{2}\right)$ is perpendicular to every vector in the tangent space we have:

$$\left(1, \frac{1}{2}, 1, -\frac{1}{2}\right) \cdot (x_1 - 1, x_2 - 2, x_3 - 1, x_4 - 6) = 0$$

Or

$$(x_1 - 1) + \frac{1}{2}(x_2 - 2) + (x_3 - 1) - \frac{1}{2}(x_4 - 6) = 0$$

Is an equation of the tangent space to M at $p = (1, 2, 1, 6)$.

Ex. Find an expression for the tangent space to $T^3 \subseteq \mathbb{R}^6$ parameterized by
$\vec{\Phi}(u_1, u_2, u_3) = (\cos u_1, \sin u_1, \cos u_2, \sin u_2, \cos u_3, \sin u_3)$
when $(u_1, u_2, u_3) = \left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right)$.

When the dimension of M is n and it's embedded in $\mathbb{R}^{n+1}$ we can write an equation for the tangent space as a single linear equation (as in the previous example). In this case, since M is not embedded in a Euclidean space of one higher dimension, so we will express the tangent space via a parametrization.

Once again $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ is a basis for $\mathbb{R}^3$:

$$\left(D\vec{\Phi}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right)\right)(1 \quad 0 \quad 0) = \vec{\Phi}_{u_1}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right)$$

$$\left(D\vec{\Phi}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right)\right)(0 \quad 1 \quad 0) = \vec{\Phi}_{u_2}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right)$$

$$\left(D\vec{\Phi}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right)\right)(1 \quad 0 \quad 0) = \vec{\Phi}_{u_3}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right).$$

where:

$$\vec{\Phi}_{u_1}(u_1, u_2, u_3) = (-\sin u_1\,, \cos u_1\,, 0, 0, 0, 0)$$
$$\vec{\Phi}_{u_1}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right) = \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0, 0, 0\right)$$

$$\vec{\Phi}_{u_2}(u_1, u_2, u_3) = (0,0, -\sin u_2\,, \cos u_2\,, 0, 0)$$
$$\vec{\Phi}_{u_2}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right) = (0, 0, -1, 0, 0, 0)$$

$$\vec{\Phi}_{u_3}(u_1, u_2, u_3) = (0, 0, 0, 0, -\sin u_3\,, \cos u_3)$$
$$\vec{\Phi}_{u_3}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right) = \left(0, 0, 0, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

So the tangent space in $\mathbb{R}^6$ must contain the point

$$\vec{\Phi}\left(\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{6}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

and vectors starting from that point given by $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0, 0, 0\right)$, $(0, 0, -1, 0, 0, 0)$, and $\left(0, 0, 0, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Thus the tangent space (3-space) to $T^3 \subseteq \mathbb{R}^6$ at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is:

$$\vec{P}(u_1, u_2, u_3) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right) + u_1\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0, 0, 0\right)$$
$$+u_2(0, 0, -1, 0, 0, 0) + u_3\left(0, 0, 0, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\vec{P}(u_1, u_2, u_3) = \left(-\frac{\sqrt{2}}{2}u_1 + \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}u_1 + \frac{\sqrt{2}}{2},\ -u_2,\ 1,\ -\frac{1}{2}u_3 + \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}u_3 + \frac{1}{2}\right)$$

$$u_1, u_2, u_3 \in \mathbb{R}.$$