Stereographic Projections of Spheres

In a homework problem (Manifolds #1) you're asked to show that the following two sets and their coordinate systems form an atlas on S^2:

$W_1 = S^2 - (0, 0, 1)$

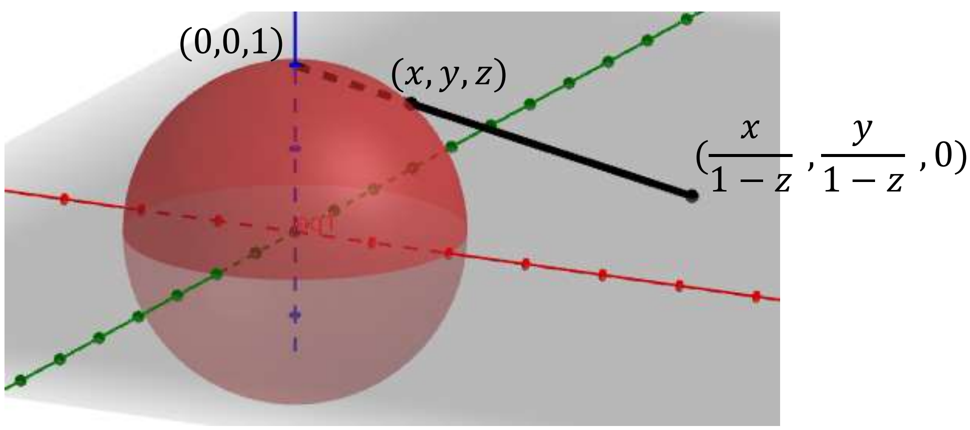
$\pi_1: W_1 \to \mathbb{R}^2$ by $\pi_1(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

$W_2 = S^2 - (0, 0, -1)$

$\pi_2: W_2 \to \mathbb{R}^2$ by $\pi_2(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$

First let's see where the mapping π_1 and π_2 come from and then generalize this approach to show that:
$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ is a manifold (this approach will also work for S^n).

Let's start with $\pi_1: S^2 - (0, 0, 1) \to \mathbb{R}^2$. Given any (x, y, z) on S^2, we can find the vector form of the line through $(0, 0, 1)$ and (x, y, z), then ask where that line intersects the xy-plane.



The direction vector of this line is given by $\vec{v} = < x, y, z - 1 >$. Since $(0, 0, 1)$ is a point on the line, an equation of the line is:

$$l(t) = < 0, 0, 1 > + t < x, y, z - 1 > = < tx, ty, t(z - 1) + 1 >$$

where $t \in \mathbb{R}$.

This line intersects the xy-plane when $t(z-1)+1=0$ or $t=\frac{1}{1-z}$.

So the point of intersection between $l(t)$ and the xy-plane is the point:

$$\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

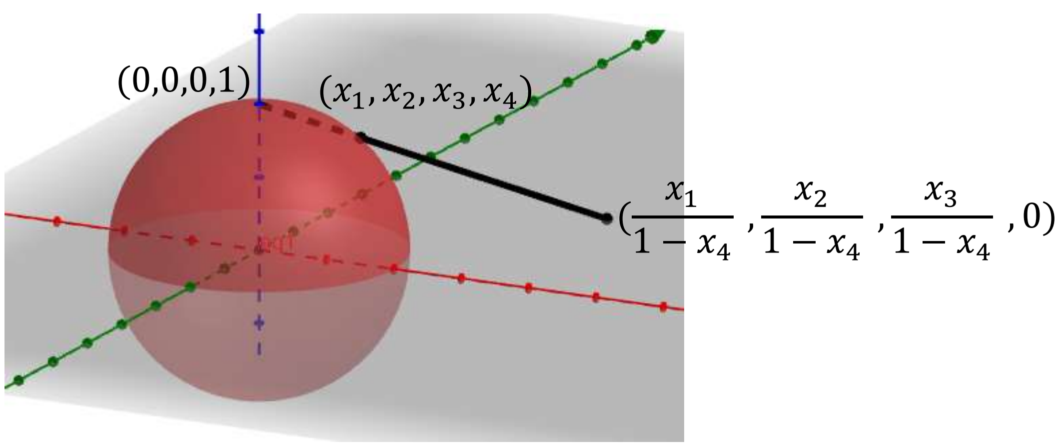
Thus $\pi_1(x,y,z)=\left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ and by a similar argument we get: $\pi_2(x,y,z)=\left(\frac{x}{1+z}, \frac{y}{1+z}\right)$. π_1 and π_2 are called **stereographic projections** of S^2 onto $\mathbb{R}^2$.

Let's take the same approach for:

$$S^3=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1^2+x_2^2+x_3^2+x_4^2=1\}$$

Let $W_1=S^3-(0,0,0,1)$ and $\pi_1: W_1\to\mathbb{R}^3$.

π_1 will take a point on $S^3-(0,0,0,1)$ and map it to the point of intersection of the line through $(0,0,0,1)$ and $(x_1,x_2,x_3,x_4)\in S^3$ with the 3-space : $\mathbb{R}^3=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_4=0\}$.



The direction vector for this line is $\vec{v} = < x_1, x_2, x_3, x_4 - 1 >$ and $(0,0,0,1)$ is a point on the line so a vector equation for the line is:

$$l(t) = < 0,0,0,1 > + t < x_1, x_2, x_3, x_4 - 1 >;$$
$$= < tx_1, tx_2, tx_3, t(x_4 - 1) + 1 > \quad ; \qquad t \in \mathbb{R}.$$

This line intersects the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\, x_4 = 0\}$ at $t(x_4 - 1) + 1 = 0$ or $t = \frac{1}{1-x_4}$. Thus, the point of intersection is:

$$\left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}, 0\right).$$

So we define π_1 by:

$$\pi_1(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4}\right).$$

Similarly, if $W_2 = S^3 - (0,0,0,-1)$ we get $\pi_2 : W_2 \to \mathbb{R}^3$ by:

$$\pi_2(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1 + x_4}, \frac{x_2}{1 + x_4}, \frac{x_3}{1 + x_4}\right).$$

How do we show π_1 and π_2 are diffeomorphisms?

Let's do this for π_1. Notice $\pi_1(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right)$ has partial derivatives of all orders because $x_4 \neq 1$ $((0,0,0,1)$ was removed from $S^3)$.

How do we know π_1 is 1-1? Suppose:

$$\pi_1(x_1, x_2, x_3, x_4) = \pi_1(w_1, w_2, w_3, w_4)$$

$$\left(\frac{x_1}{1-x_4}, \frac{x_2}{1-x_4}, \frac{x_3}{1-x_4}\right) = \left(\frac{w_1}{1-w_4}, \frac{w_2}{1-w_4}, \frac{w_3}{1-w_4}\right)$$

or

$$(*) \qquad \frac{x_1}{1-x_4} = \frac{w_1}{1-w_4}; \quad \frac{x_2}{1-x_4} = \frac{w_2}{1-w_4}; \quad \frac{x_1}{1-x_4} = \frac{w_3}{1-w_4}.$$

But notice:

$$\left(\frac{x_1}{1-x_4}\right)^2 + \left(\frac{x_2}{1-x_4}\right)^2 + \left(\frac{x_3}{1-x_4}\right)^2 + 1 = \frac{x_1^2+x_2^2+x_3^2+(1-x_4)^2}{(1-x_4)^2}$$

$$= \frac{x_1^2+x_2^2+x_3^2+x_4^2+1-2x_4}{(1-x_4)^2}$$

$$= \frac{2(1-x_4)}{(1-x_4)^2} = \frac{2}{1-x_4}$$

By the same argument:

$$\left(\frac{w_1}{1-w_4}\right)^2 + \left(\frac{w_2}{1-w_4}\right)^2 + \left(\frac{w_3}{1-w_4}\right)^2 + 1 = \frac{2}{1-w_4}$$

But by $(*)$ that means:

$$\frac{2}{1-x_4} = \frac{2}{1-w_4} \qquad \Rightarrow \; 1 - x_4 = 1 - w_4$$

$$x_4 = w_4 \,.$$

Using $(*)$ again we get:

$$x_1 = w_1; \quad x_2 = w_2; \quad x_3 = w_3; \quad x_4 = w_4 \,.$$

Thus π_1 is 1-1.

How do we know π_1 is onto?

In answering this question we will actually find π_1^{-1}. That is, given any $(a, b, c) \in \mathbb{R}^3$, how do we find x_1, x_2, x_3, x_4 such that:

$$\pi_1(x_1, x_2, x_3, x_4) = (a, b, c)$$

$$\frac{x_1}{1-x_4} = a; \qquad \frac{x_2}{1-x_4} = b; \qquad \frac{x_3}{1-x_4} = c.$$

As before:

$$\frac{2}{1-x_4} = \left(\frac{x_1}{1-x_4}\right)^2 + \left(\frac{x_2}{1-x_4}\right)^2 + \left(\frac{x_3}{1-x_4}\right)^2 + 1 = a^2 + b^2 + c^2 + 1$$

$$\frac{1-x_4}{2} = \frac{1}{a^2+b^2+c^2+1}$$

$$1 - x_4 = \frac{2}{a^2+b^2+c^2+1}.$$

Thus we can write:

$$x_1 = a(1 - x_4) = \frac{2a}{a^2+b^2+c^2+1}$$

$$x_2 = b(1 - x_4) = \frac{2b}{a^2+b^2+c^2+1}$$

$$x_3 = c(1 - x_4) = \frac{2c}{a^2+b^2+c^2+1}$$

$$x_4 = 1 - \frac{2}{a^2+b^2+c^2+1} = \frac{a^2+b^2+c^2-1}{a^2+b^2+c^2+1}.$$

Thus:

$$\pi_1\left(\frac{2a}{a^2+b^2+c^2+1},\frac{2b}{a^2+b^2+c^2+1},\frac{2c}{a^2+b^2+c^2+1},\frac{a^2+b^2+c^2-1}{a^2+b^2+c^2+1}\right)=(a,b,c).$$

and π_1 is onto.

What's more we just showed:

$$\pi_1^{-1}(u_1,u_2,u_3)=\left(\frac{2u_1}{u_1^2+u_2^2+u_3^2+1},\frac{2u_2}{u_1^2+u_2^2+u_3^2+1},\frac{2u_3}{u_1^2+u_2^2+u_3^2+1},\frac{u_1^2+u_2^2+u_3^2-1}{u_1^2+u_2^2+u_3^2+1}\right)$$

Notice also that π_1^{-1} has partial derivatives at all order for any (u_1,u_2,u_3). Hence, π_1 is a diffeomorphism.

Similar arguments show that π_2 is also a diffeomorphism.

How do we know $\pi_1^{-1}(\mathbb{R}^3)\cup\pi_2^{-1}(\mathbb{R}^3)\supseteq S^3$?

$$\pi_1^{-1}(\mathbb{R}^3)=S^3-(0,0,0,1)$$
$$\pi_2^{-1}(\mathbb{R}^3)=S^3-(0,0,0,-1)$$

Since $(0,0,0,-1)\in S^3-(0,0,0,1)$ we have:

$$\pi_1^{-1}(\mathbb{R}^3)\cup\pi_2^{-1}(\mathbb{R}^3)\supseteq S^3.$$

Let's find the transition function $\pi_2\pi_1^{-1}(u_1, u_2, u_3)$ and show $\{\pi_i, W_i\}$ is a smooth atlas for S^3.

$$\pi_2\pi_1^{-1}(u_1, u_2, u_3) = \pi_2\left(\frac{2u_1}{u_1^2+u_2^2+u_3^2+1}, \frac{2u_2}{u_1^2+u_2^2+u_3^2+1}, \frac{2u_3}{u_1^2+u_2^2+u_3^2+1}, \frac{u_1^2+u_2^2+u_3^2-1}{u_1^2+u_2^2+u_3^2+1}\right)$$

and

$$\pi_2(x_1, x_2, x_3, x_4) = \left(\frac{x_1}{1+x_4}, \frac{x_2}{1+x_4}, \frac{x_3}{1+x_4}\right).$$

Notice:

$$1 + x_4 = 1 + \frac{u_1^2+u_2^2+u_3^2-1}{u_1^2+u_2^2+u_3^2+1} = \frac{2(u_1^2+u_2^2+u_3^2)}{u_1^2+u_2^2+u_3^2+1}$$

so

$$\pi_2\pi_1^{-1}(u_1, u_2, u_3) = \left(\frac{u_1}{u_1^2+u_2^2+u_3^2}, \frac{u_2}{u_1^2+u_2^2+u_3^2}, \frac{u_3}{u_1^2+u_2^2+u_3^2}\right)$$

where W_1 intersects W_2 i.e. on $\pi_1(W_1 \cap W_2) = \mathbb{R}^3 - (0, 0, 0)$.

Thus, $\pi_2\pi_1^{-1}$ has partial derivatives of all orders.