# Manifolds

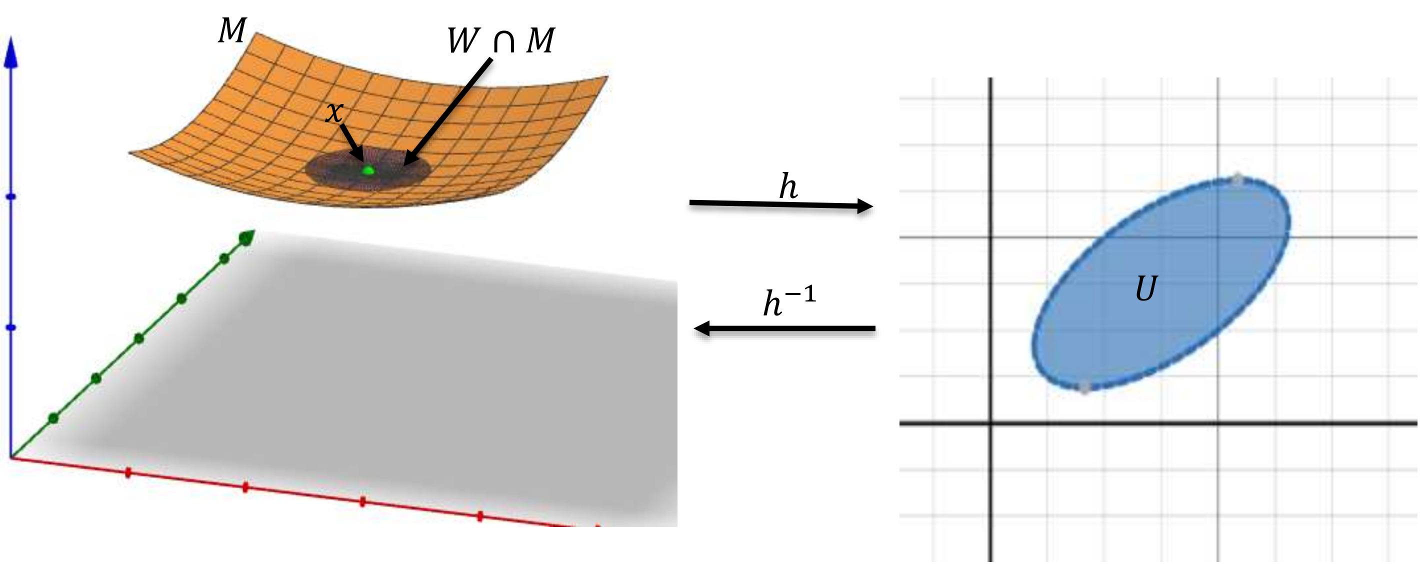
Def. Let $U$ and $V$ be open sets in $\mathbb{R}^n$. A differentiable function, $h: U \to V$ with a differentiable inverse $h^{-1}: V \to U$, is called a **diffeomorphism** ("differentiable" will mean $C^\infty$ from here on).

Def. A subset, $M \subseteq \mathbb{R}^n$, is called a **differentiable manifold** (or just a manifold) of dimension $k$ if for each point $x \in M$ there is an open set $W \subseteq \mathbb{R}^n$, an open set $U \subseteq \mathbb{R}^k$, and a diffeomorphism:

$$h: W \cap M \to U.$$

$h$ is called a **system of coordinates** on $W \cap M$.

$h^{-1}: U \to W \cap M$ is called a **parameterization** of $W \cap M$.



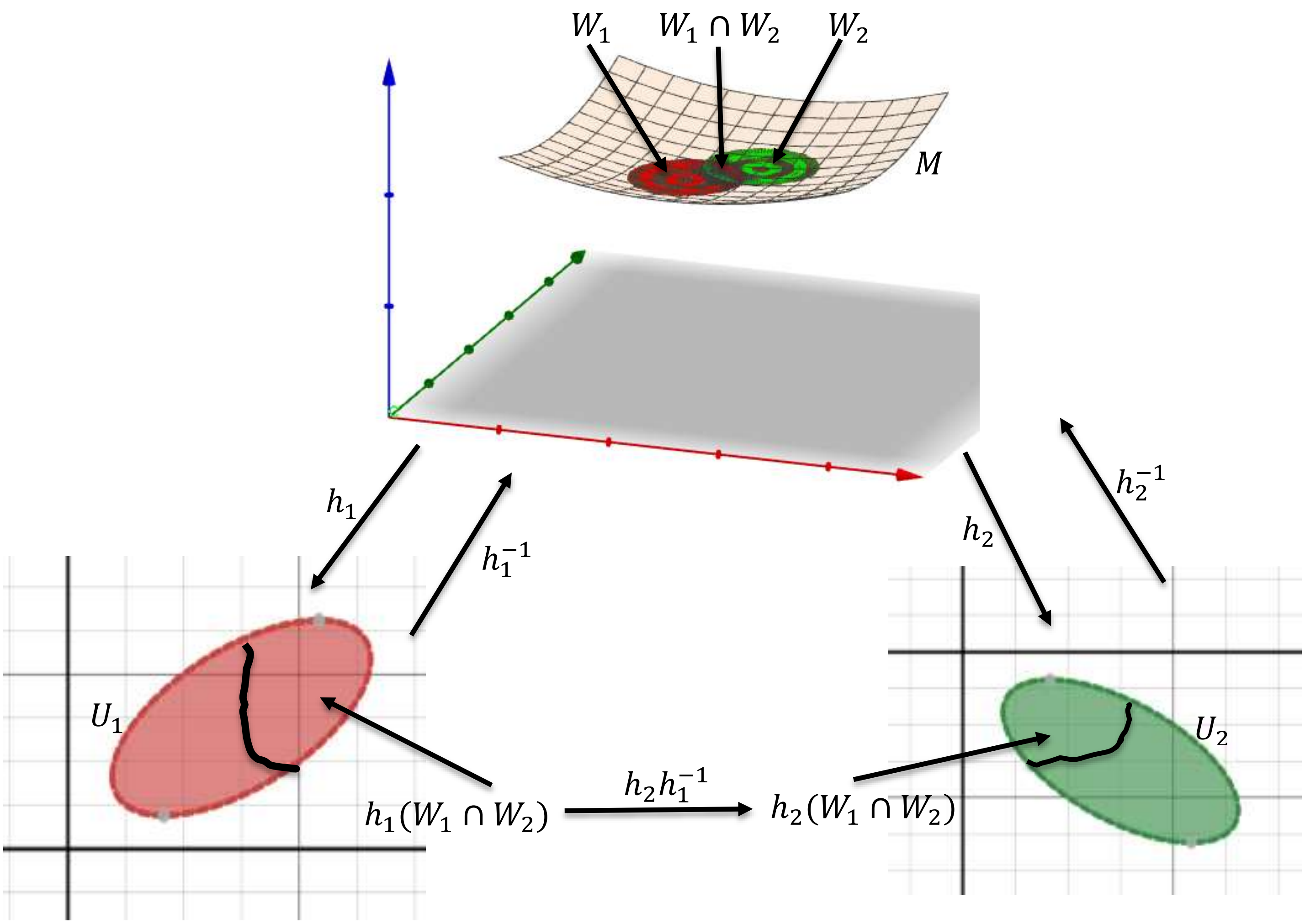

The set $\{h_\alpha, W_\alpha\}$ of coordinate functions and sets $W_\alpha$ that cover $M$ is called an **atlas**.

Ex. A point in $\mathbb{R}^n$ is a zero dimensional manifold.
An open set in $\mathbb{R}^n$ is an $n$-dimensional manifold.

Notice that if $(h_1, W_1)$ and $(h_2, W_2)$ are two coordinate systems on $W_1, W_2 \subseteq M$, where $h_1 : W_1 \to U_1$ and $h_2 : W_2 \to U_2$, then:

$$h_{12} = h_2 h_1^{-1} : h_1(W_1 \cap W_2) \to h_2(W_1 \cap W_2)$$

*is a differentiable map of an open set in* $\mathbb{R}^k$ *into an open set in* $\mathbb{R}^k$, and is called a **transition function** between the coordinate systems $(h_1, W_1)$ and $(h_2, W_2)$.



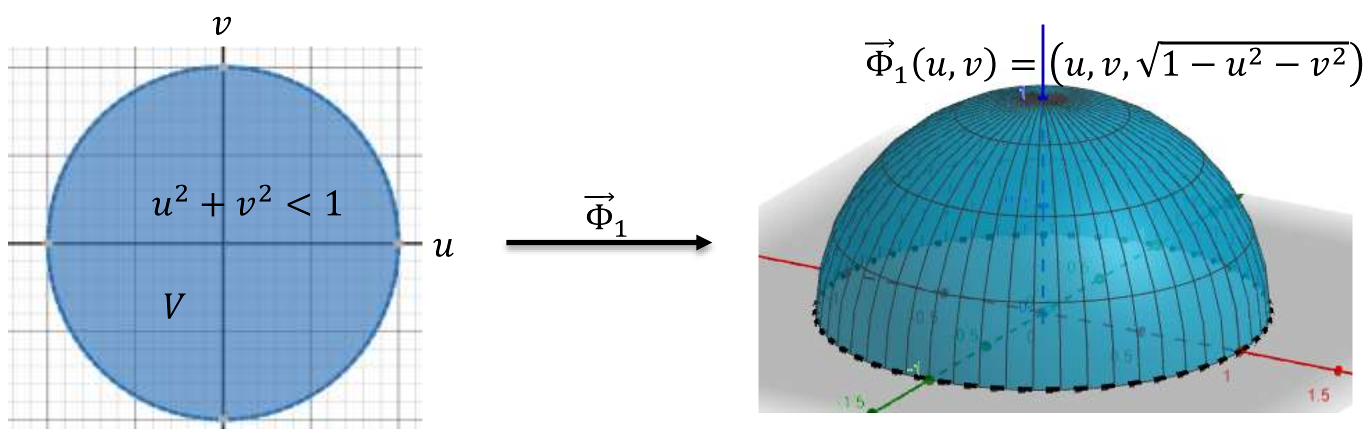

Def. An atlas $(h_\alpha, W_\alpha)$ is called **smooth** if all of the transition functions are smooth.

Ex. Show that $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ is a (differentiable) manifold.

One way to do this is to define the following 6 parameterizations of the sphere, which cover the entire sphere.

$\vec{\Phi}_i: V \to \mathbb{R}^3$ where $V = \{(u, v) \in \mathbb{R}^2 | \, u^2 + v^2 < 1\}$

$$\vec{\Phi}_1(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right) \qquad (z > 0)$$
$$\vec{\Phi}_2(u, v) = \left(u, v, -\sqrt{1 - u^2 - v^2}\right) \qquad (z < 0)$$
$$\vec{\Phi}_3(u, v) = \left(u, \sqrt{1 - u^2 - v^2}, v\right) \qquad (y > 0)$$
$$\vec{\Phi}_4(u, v) = \left(u, -\sqrt{1 - u^2 - v^2}, v\right) \qquad (y < 0)$$
$$\vec{\Phi}_5(u, v) = \left(\sqrt{1 - u^2 - v^2}, u, v\right) \qquad (x > 0)$$
$$\vec{\Phi}_6(u, v) = \left(-\sqrt{1 - u^2 - v^2}, u, v\right) \qquad (x < 0)$$




To show that these 6 parameterizations make $S^2$ into a manifold we must show:

1) $\vec{\Phi}_i$ is a diffeomorphism, for $i = 1, \ldots, 6$
2) $\bigcup_{i=1}^{6} \vec{\Phi}_i(V) \supseteq S^2$.

To show that $\overrightarrow{\Phi}_i$ is a diffeomorphism we must show:

a. $\overrightarrow{\Phi}_i$ is one to one
b. $\overrightarrow{\Phi}_i$ is onto its image
c. $\overrightarrow{\Phi}_i$ and $\overrightarrow{\Phi}_i^{-1}$ are differentiable.

Let's show that $\overrightarrow{\Phi}_1$ is a diffeomorphism.

a.
$$\overrightarrow{\Phi}_1(u,v) = \overrightarrow{\Phi}_1(u',v')$$
$$\left(u, v, \sqrt{1-u^2-v^2}\right) = \left(u', v', \sqrt{1-u'^2-v'^2}\right)$$
So $(u,v) = (u',v')$ and $\overrightarrow{\Phi}_1$ is one to on.

b. By definition $\overrightarrow{\Phi}_1$ maps $V$ onto $\overrightarrow{\Phi}_1(V)$.

c. Each $\overrightarrow{\Phi}_i$ is differentiable on $V$ because all of the partial derivatives of all order exist (since $u^2 + v^2 \neq 1$). The inverse functions of the $\overrightarrow{\Phi}_i$s are just projections. For example:

$$\left(\overrightarrow{\Phi}_1\right)^{-1}\left(u, v, \sqrt{1-u^2-v^2}\right) = (u,v).$$

All partial derivatives of all orders exist so $\left(\overrightarrow{\Phi}_1\right)^{-1}$ is differentiable. The same holds for the other $\left(\overrightarrow{\Phi}_i\right)^{-1}$.

$\bigcup_{i=1}^{6} \overrightarrow{\Phi}_i\,(V) \supseteq S^2$ because every point of $S^2$ has at least one non-zero coordinate.

What do the transition functions look like? First, notice that not all $\vec{\Phi}_i(V), \vec{\Phi}_j(V)$ intersect (e.g. $\vec{\Phi}_1(V)$ is the upper hemisphere and $\vec{\Phi}_2(V)$ is the lower hemisphere). As an example, let's look at $\vec{\Phi}_1(V) \cap \vec{\Phi}_3(V)$.

$\vec{\Phi}_1(V) =$ points on $S^2$ with $z > 0$
$\vec{\Phi}_3(V) =$ points on $S^2$ with $y > 0$

$\vec{\Phi}_1(V) \cap \vec{\Phi}_3(V) =$ points on $S^2$ with $y > 0$ and $z > 0$.

$\vec{\Phi}_3(u,v) = \left(u, \sqrt{1-u^2-v^2}, v\right)$
$\vec{\Phi}_3^{-1}\left(u, \sqrt{1-u^2-v^2}, v\right) = (u,v)$.

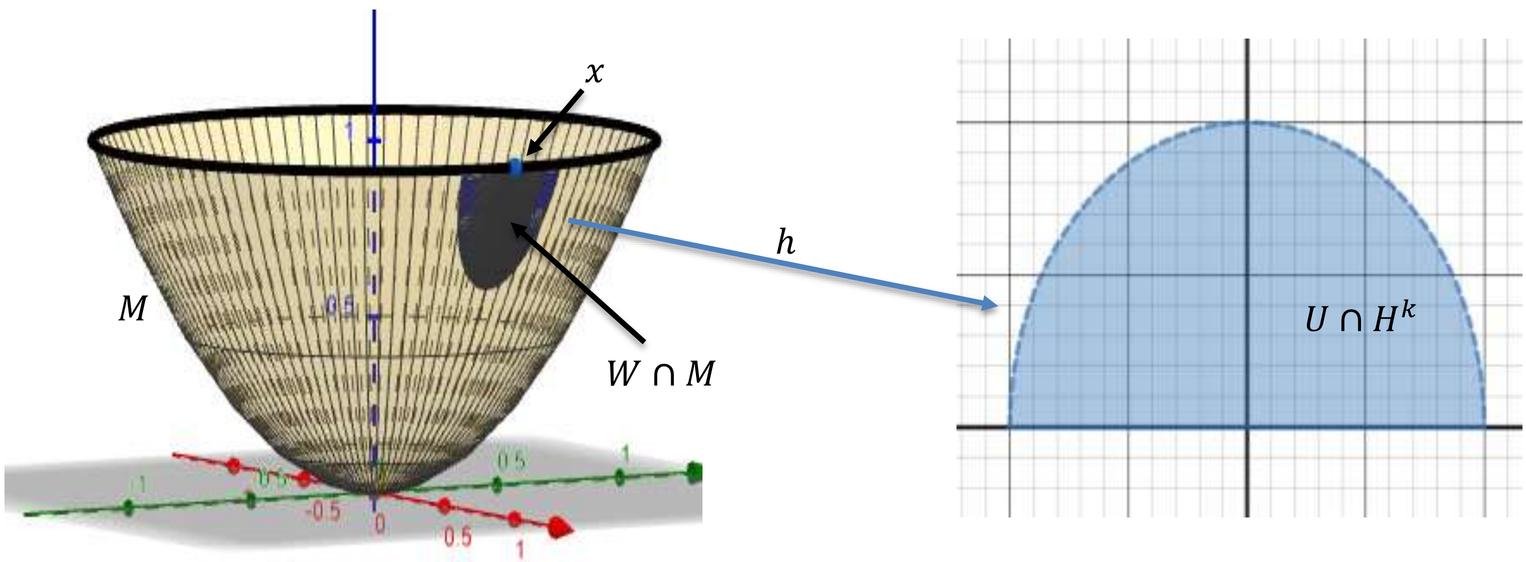
So $\left(\vec{\Phi}_3\right)^{-1}\vec{\Phi}_1(u,v) = \vec{\Phi}_3^{-1}\left(u, v, \sqrt{1-u^2-v^2}\right) = \left(u, \sqrt{1-u^2-v^2}\right)$.

Other transition functions are also differentiable, thus $\left\{\vec{\Phi}_i^{-1}, \vec{\Phi}_i(V)\right\}$ for $i = 1, \ldots, 6$ is a smooth atlas for $S^2$.

Def. $H^k = \{x \in \mathbb{R}^k \,|\, x_k \geq 0\}$, is called the **half-space**.

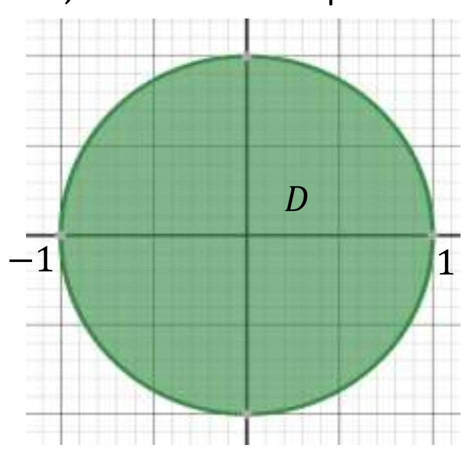
Ex. $H^2$ is the upper half plane including the $x$-axis.
$H^3 = \{(x,y,z) \in \mathbb{R}^3 | z \geq 0\}$.

Def. $M \subseteq \mathbb{R}^n$ is a ***k*-dimensional manifold with boundary** if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open set $U \subseteq \mathbb{R}^k$ or diffeormorphic to $U \cap H^k$, where $U$ is an open set in $\mathbb{R}^k$. The set of points in $M$ where $W \cap M$ is diffeomorphic to $U \cap H^k$ are called **boundary points** of $M$.

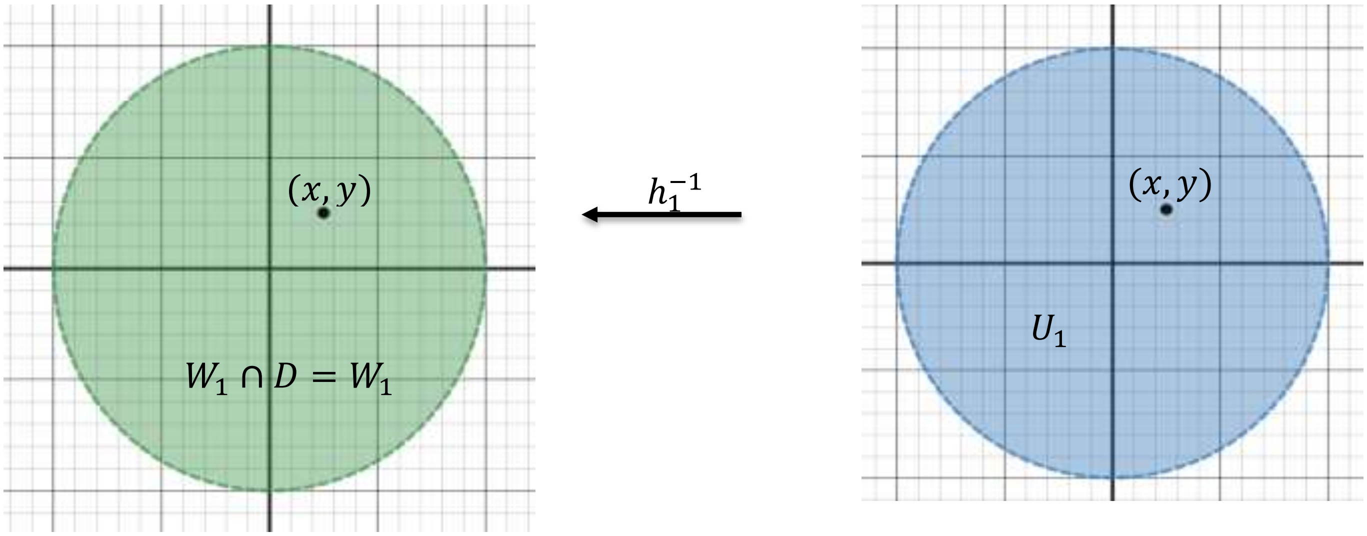



Ex. Show that the closed unit disk, $D = \{(x, y) \in \mathbb{R}^2 | \; x^2 + y^2 \leq 1\}$, is a manifold with boundary.

We need to show that for each point $(x, y) \in D$, there is a open set $W \subseteq \mathbb{R}^2$ containing $(x, y)$ such that $W \cap D$ is diffeomorphic to an open set $U \subseteq \mathbb{R}^2$ or diffeomorphic to $U \cap H^2$, where $U$ is an open set in $\mathbb{R}^2$.



Notice for points $(x, y) \in D$ such that $x^2 + y^2 < 1$ this is easy to do.




For these points let:

$$U_1 = W_1 = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 < 1\}, \text{ and let}$$

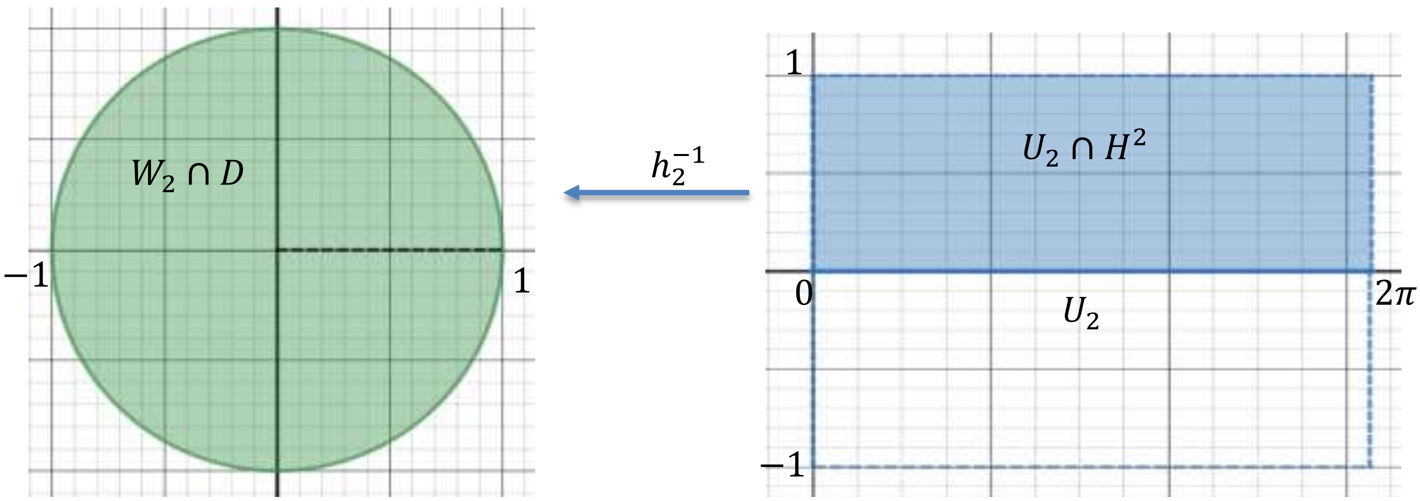
$$h_1^{-1}: U_1 \subseteq \mathbb{R}^2 \rightarrow W_1 \cap D = W_1 \quad \text{by}$$
$$h_1^{-1}(x, y) = (x, y)$$

$h_1^{-1}$ is the identity function and is clearly one-one, onto, and is its own inverse. Also, $h_1^{-1}$ and $h_1$ are differentiable. Thus $h_1^{-1}$ is a diffeomorphism.

To cover points on the boundary of $D$ we need to do more work.
We'll need 2 more sets to do this. We need to find open sets $U$ and $W$ such that $U \cap H^2$ is diffeomorphic to $W \cap D$.

Let $U_2 = \{(x,y)|\ 0 < x < 2\pi,\ -1 < y < 1\}$
$W_2 = \mathbb{R}^2 - \{(x,0) \in \mathbb{R}^2|\ x \geq 0\}$.

Then $U_2 \cap H^2 = \{(x,y)|\ 0 < x < 2\pi,\ 0 \leq y < 1\}$.
$W_2 \cap D = D - \{(x,0)|\ 0 \leq x \leq 1\}$.




Now define: $h_2^{-1}: U_2 \cap H^2 \to W_2 \cap D$ by
$$h_2^{-1}(x,y) = \big((1-y)cosx, (1-y)sinx\big).$$

Notice that for each fixed $y$, $0 \leq y < 1$, $h_2^{-1}$ maps the open interval $(x,y)$, $0 < x < 2\pi$, onto a circle of radius $1 - y$, centered at $(0,0)$ minus a point on the positive $x$-axis.

Now we need to show that $h_2^{-1}$ is a diffeomorphism.

Claim: $h_2^{-1}$ is one to one.

Suppose $h_2^{-1}(x_1, y_1) = h_2^{-1}(x_2, y_2)$, $0 < x_1, x_2 < 2\pi$, $0 \leq y_1, y_2 < 1$
Then:

$$(1 - y_1)cosx_1 = (1 - y_2)cosx_2$$
$$(1 - y_1)sinx_1 = (1 - y_2)sinx_2$$

Now square both equations and add them:

$$(1 - y_1)^2 \cos^2 x_1 + (1 - y_1)^2 \sin^2 x_1 = (1 - y_2)^2 \cos^2 x_2 + (1 - y_2)^2 \sin^2 x_2 .$$

Thus we have:

$$(1 - y_1)^2 = (1 - y_2)^2; \qquad 0 \leq y_1, y_2 < 1$$

So:

$$y_1 = y_2.$$

Since $y_1 = y_2$, and $1 - y_1 \neq 0$, we can divide the original 2 equations by $1 - y_1$.

$cosx_1 = cosx_2$ so $x_2 = x_1$ or $x_2 = 2\pi - x_1$
$sinx_1 = sinx_2$ so $x_2 = x_1$ or $x_2 = \pi - x_1$.

Hence $x_1 = x_2$, and $h_2^{-1}$ is one to one.

To show that $h_2^{-1}$ is onto $W_2 \cap D$ we'll show that given any point in $W_2 \cap D$ we can find a point in $U_2 \cap H^2$ that maps onto it. That is, we will find the inverse function, $h_2$.

To do this we need to solve $x = x(u, v)$, $y = y(u, v)$ in:

$$u = (1 - y)cosx$$
$$v = (1 - y)sinx.$$

Squaring the 2 equations and adding we get:

$$u^2 + v^2 = (1-y)^2 \cos^2 x + (1-y)^2 \sin^2 x = (1-y)^2.$$

Since $1 - y > 0$, we only get one square root above:

$$1 - y = \sqrt{u^2 + v^2}$$

or $$y = 1 - \sqrt{u^2 + v^2}.$$

Notice that all of the partial derivatives of $y$ of all orders exist since $(u, v) \neq (0,0)$.

Since $1 - y > 0$, we have $1 - y \neq 0$, so we can divide the 2 original equations

$$\frac{v}{u} = tanx.$$

For the set $U_2 \cap H^2$, $0 < x < 2\pi$, so we need to define the inverse of the above equation carefully:

$$x = \tan^{-1}\frac{v}{u} \quad \text{if } (u,v) \text{ is in the 1st quadrant}$$
$$= \frac{\pi}{2} \quad \text{if } (u,v) = (0,1)$$
$$= \pi + \tan^{-1}\frac{v}{u} \quad \text{if } (u,v) \text{ is in the 2nd/3rd quadrant}$$
$$= \frac{3\pi}{2} \quad \text{if } (u,v) = (0,-1)$$
$$= 2\pi + \tan^{-1}\frac{v}{u} \quad \text{if } (u,v) \text{ is in the 4th quadrant.}$$

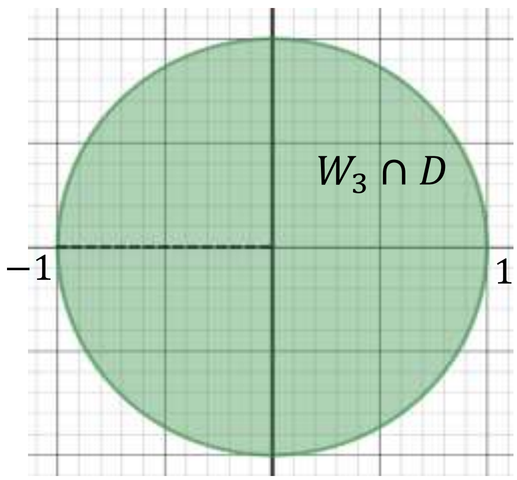
It's not hard to show that all partial derivatives of all orders exist for $x$ since $0 < x < 2\pi$.

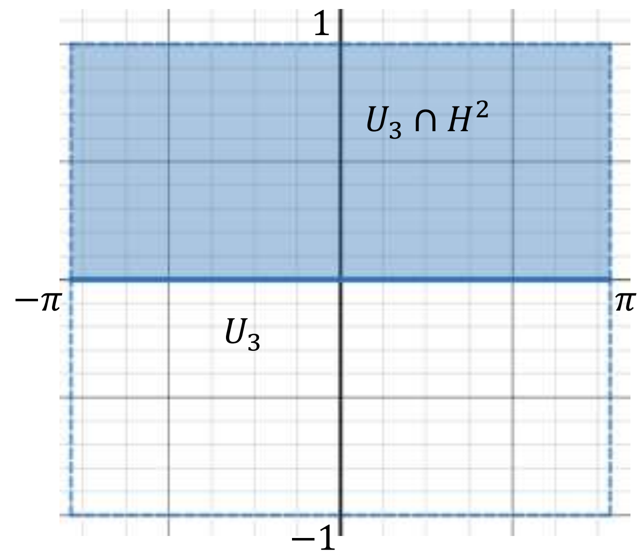
Thus if we say $x(u, v)$ is the complicated formula written above and $y(u,v) = 1 - \sqrt{u^2 + v^2}$, then $h_2(u, v) = (x(u, v), y(u, v))$ is the differentiable inverse of $h_2^{-1}(x, y)$.

$h_2^{-1}$ is clearly differentiable, thus, $h_2^{-1}$ is a diffeomorphism.

Finally, let $U_3 = \{(x,y) | -\pi < x < \pi,\ -1 < y < 1\}$
$W_3 = \mathbb{R}^2 - \{(x,0) \in \mathbb{R}^2 |\ x \leq 0\}.$

Then $U_3 \cap H^2 = \{(x,y) | -\pi < x < \pi,\ 0 \leq y < 1\}.$
$W_3 \cap D = D - \{(x,0) | -1 \leq x \leq 0\}.$




Now define: $h_3^{-1}: U_3 \cap H^2 \to W_3 \cap D$ by
$$h_3^{-1}(x,y) = \big((1-y)cosx, (1-y)sinx\big).$$

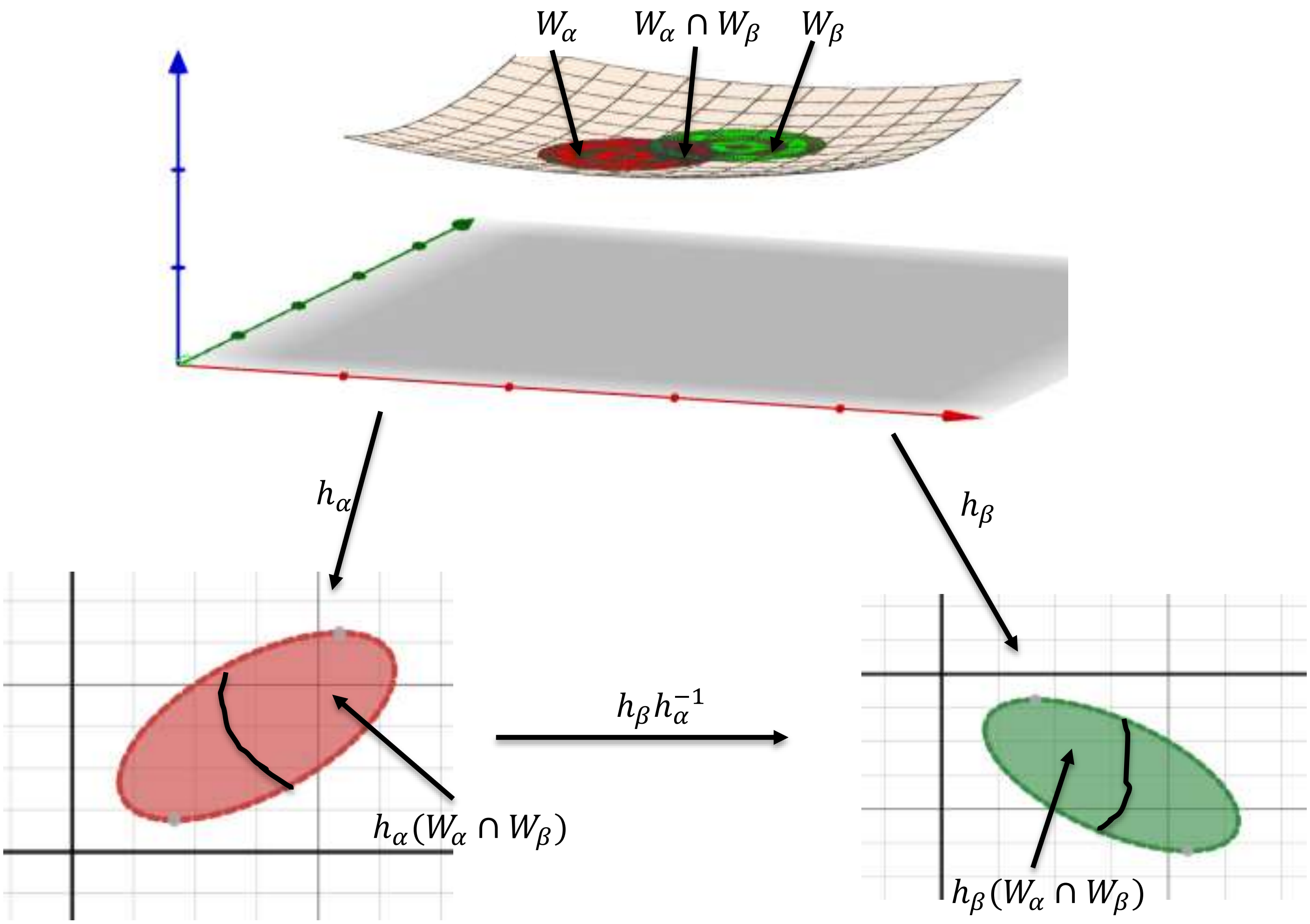
A similar argument to the one used to show $h_2^{-1}$ is a diffeomorphism shows that $h_3^{-1}$ is a diffeomorpism.

Now note that: $h_2^{-1}(U_2) \cup h_3^{-1}(U_3) = D - (0,0)$,
but $(0,0) \in h_1^{-1}(U_1)$.

Thus we have:

$\bigcup_{i=1}^{3} h_i(U_i) \supseteq D$, and $D$ is a differentiable manifold with boundary.

Def. Let $M$ be a differentiable manifold of dimension $k$. We say $M$ is orientable if there is an atlas for $M$, $\{h_\alpha, W_\alpha\}$, such that all of the transition functions: $h_\beta \circ h_\alpha^{-1}: h_\alpha(W_\alpha \cap W_\beta) \to h_\beta(W_\alpha \cap W_\beta)$ have positive Jacobians (i.e. $\det\left(\left(h_\beta \circ h_\alpha^{-1}\right)'\right) > 0$).



Ex. Consider the following atlas on $S^2$

$$\pi_1 : S^2 - (0,0,1) \to \mathbb{R}^2$$

$$\pi_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$\pi_2 : S^2 - (0,0,-1) \to \mathbb{R}^2$$

$$\pi_2(x,y,z) = \left(\frac{x}{1+z}, -\frac{y}{1+z}\right).$$

From a homework problem you will see that:

$$\pi_1^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

Thus we have:

$$(\pi_2 \circ \pi_1^{-1})(u,v) = \pi_2\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) = \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2}\right)$$

$$(\pi_2 \circ \pi_1^{-1})'(u,v) = \begin{pmatrix} \frac{u^2-v^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} \\ \frac{2uv}{(u^2+v^2)^2} & \frac{u^2-v^2}{(u^2+v^2)^2} \end{pmatrix}$$

$$\det((\pi_2 \circ \pi_1^{-1})') = \frac{(u^2-v^2)^2+4u^2v^2}{(u^2+v^2)^4} = \frac{1}{(u^2+v^2)^2} > 0$$

and finite since $\pi_1^{-1}(0,0) = (0,0,-1)$, which is not part of the domain of $\pi_2$. Thus we can say $S^2$ is orientable.

Note: the atlas with $\pi_1(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ and $\pi_2(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$; the standard stereographic projection does not have:

$$\det((\pi_2 \circ \pi_1^{-1})') > 0.$$