Manifolds- HW Problems

1. Show that the unit sphere, $S^2 \subseteq \mathbb{R}^3$, is a manifold by covering it with the following two sets and their coordinate systems:
   $W_1 = S^2 - (0,0,1)$; and
   $\pi_1: W_1 \to \mathbb{R}^2$ by $\pi_1(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z})$
   $W_2 = S^2 - (0,0,-1)$; and
   $\pi_2: W_2 \to \mathbb{R}^2$ by $\pi_2(x,y,z) = (\frac{x}{1+z}, \frac{y}{1+z})$.
   π_1 and π_2 are called stereographic projections of S^2.

   a. Show that π_1 and π_2 are diffeomorphisms by
      i. Showing that π_1 and π_2 have partial derivatives of all orders.
      ii. Showing π_1 and π_2 are 1-1 and onto $\mathbb{R}^2$. Hint: for both 1-1 and onto consider the following:
      $$\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 + 1 = \frac{x^2+y^2+(1-z)^2}{(1-z)^2} = \frac{2}{1-z}$$
      $$\left(\frac{x}{1+z}\right)^2 + \left(\frac{y}{1+z}\right)^2 + 1 = \frac{x^2+y^2+(1+z)^2}{(1+z)^2} = \frac{2}{1+z}$$
      iii. Showing that $\pi_1^{-1}(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$
      and $\pi_2^{-1}(u,v) = (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1})$
      iv. Showing that π_1^{-1} and π_2^{-1} have partial derivatives of all orders.

   b. Show that $\pi_1^{-1}(\mathbb{R}^2) \cup \pi_2^{-1}(\mathbb{R}^2) \supseteq S^2$.
   c. Find the transition function $\pi_2\pi_1^{-1}(u,v)$ and show that $\{\pi_i, W_i\}$ is a smooth atlas for S^2.

2. Let $S^1 = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 = 1\}$. Let

   $U_1 = \{(x, y) \in S^1 | \, x > 0\}$, $\quad U_2 = \{(x, y) \in S^1 | \, x < 0\}$,

   $U_3 = \{(x, y) \in S^1 | \, y > 0\}$, $\quad U_4 = \{(x, y) \in S^1 | \, y < 0\}$.

   Define the functions:

   $h_1: U_1 \to \mathbb{R}$ by $h_1(x, y) = y$, $\quad h_2: U_2 \to \mathbb{R}$ by $h_2(x, y) = y$

   $h_3: U_3 \to \mathbb{R}$ by $h_3(x, y) = x$, $\quad h_4: U_4 \to \mathbb{R}$ by $h_4(x, y) = x$.

   Notice, for example, $U_1 = \{(x, y) \in \mathbb{R}^2 | \, x = \sqrt{1 - y^2}, -1 < y < 1\}$,

   so $h_1(x, y) = h_1\left(\sqrt{1 - y^2}, y\right) = y$.

   a. Show that the h_i are diffeomorphisms.
   b. Show that $\bigcup_{i=1}^{4}(U_i) \supseteq S^1$.
   c. Show that $\{h_i, U_i\}$ is a smooth atlas for S^1.