Integration over Singular n-chains and Stokes' Theorem

If ω is a k-form on $[0,1]^k$, then $\omega = f(x_1, \dots, x_k)dx_1 \wedge \dots \wedge dx_k$ for a unique function, $f: [0,1]^k \to \mathbb{R}$.

We have already defined the Riemann integral of f over $[0,1]^k$, $\int_{[0,1]^k} f$. We now define:

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f dx_1 \wedge \dots \wedge dx_k = \int_{[0,1]^k} f$$

Note on notation:

$$\int_{[0,1]^k} f dx_1 \wedge \dots \wedge dx_k = \int_{[0,1]^k} f dx_1 \dots dx_k$$

Ex. Let $\omega = (x+y)dx \wedge dy$ on $[0,1] \times [0,1]$

$$\int_{[0,1]^2} \omega = \int_0^1 \int_0^1 (x+y)\, dx\, dy$$

We can evaluate this with Fubini's Theorem.

Def. If ω is a k-form on $A \subseteq \mathbb{R}^n$ and c is a singular k-cube in A, then we define:

$$\int_{\boldsymbol{c}} \boldsymbol{\omega} = \int_{[\mathbf{0},\mathbf{1}]^{\boldsymbol{k}}} \boldsymbol{c}^*(\boldsymbol{\omega}).$$

Note: We have defined a singular k-cube as a map $c: [0, 1]^k \to \mathbb{R}^n$. So if we have a k-form, ω, on $A \subseteq \mathbb{R}^n$, then we define:

$$\int_c \omega = \int_{[0,1]^k} c^*(\omega).$$

However, we don't need c to map $[0, 1]^k$ into $\mathbb{R}^n$. We can make the same definition for integration for $c': D \to \mathbb{R}^n$, where $D \subseteq \mathbb{R}^k$ and D is compact (but D is not necessarily $[0, 1]^k$).

We define:

$$\int_{\boldsymbol{c'}} \boldsymbol{\omega} = \int_{\boldsymbol{D}} (\boldsymbol{c'})^*(\boldsymbol{\omega})$$

Since $D \subseteq \mathbb{R}^k$ and compact, we can still use Fubini's Theorem to evaluate:

$$\int_D (c')^*(\omega).$$

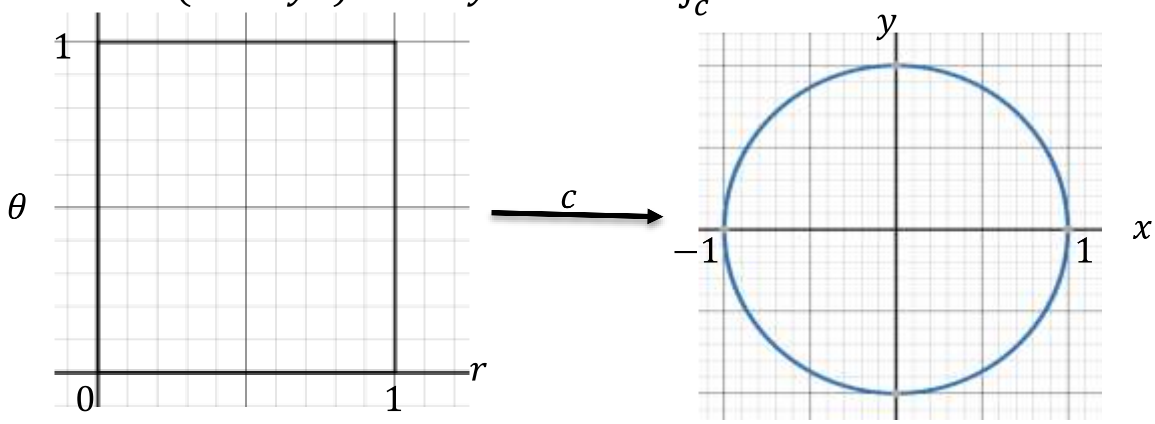
If c is a k-chain, then we write:

$$c = \sum_{i=1}^{j} a_i c_i$$

$$\int_c \omega = \sum_{i=1}^{j} a_i \int_{c_i} \omega.$$

Ex. Let $c: [0,1]^2 \to \mathbb{R}^2$ by $c(r,\theta) = (r\cos 2\pi\theta\,, r\sin 2\pi\theta)$, so c maps the square $[0,1]^2$ into the unit disk, D, in $\mathbb{R}^2$.

Let $\omega = (x^2 + y^2)dx \wedge dy$ on D. Find $\int_c \omega$.



$$\int_c \omega = \int_{[0,1]^2} c^*\omega = \int_{[0,1]^2} c^*\big((x^2 + y^2)dx \wedge dy\big)$$

$$c^*\big((x^2 + y^2)dx \wedge dy\big) = [(x^2 + y^2) \circ c]\; c^*(dx) \wedge c^*(dy)$$

$$= (r^2\cos^2 2\pi\theta + \; r^2\sin^2 2\pi\theta)\;\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) \wedge \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$

$$c^*(\omega) = r^2[(\cos 2\pi\theta)dr - (2\pi r\sin 2\pi\theta)d\theta] \\ \wedge [(\sin 2\pi\theta)dr + (2\pi r\cos 2\pi\theta)d\theta]$$

$$= r^2[-2\pi r\sin^2(2\pi\theta)\; d\theta \wedge dr + (2\pi r\cos^2(2\pi\theta))dr \wedge d\theta$$

$$= r^2(2\pi r(\sin^2(2\pi\theta) + \cos^2(2\pi\theta))dr \wedge d\theta$$

$$= 2\pi r^3 dr \wedge d\theta.$$

$$\int_c \omega = \int_{[0,1]^2} 2\pi r^3 dr \wedge d\theta = \int_0^1 \int_0^1 2\pi r^3 dr\, d\theta$$

$$= \int_0^1 \left.\frac{\pi r^4}{2}\right|_0^1 d\theta = \int_0^1 \frac{\pi}{2}\, d\theta = \frac{\pi}{2}.$$

In a 2nd year calculus course, here's how we would have evaluated the same integral:

$$\iint_D (x^2 + y^2) dx\, dy\,.$$

We would then change to polar coordinates, so

$$x^2 + y^2 = r^2 \quad \text{and} \quad dx\, dy = r\, dr\, d\theta.$$

$$\iint_D (x^2 + y^2) dx\, dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2) r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left.\frac{r^4}{4}\right|_0^1 d\theta = \int_{\theta=0}^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}.$$

Notice that if we had let $c': [0,1] \times [0,2\pi] \to \mathbb{R}^2$ by $c'(r,\theta) = (r\cos\theta\,, r\sin\theta)$, then

$$(c')^*\big((x^2+y^2)dx \wedge dy\big) = [(x^2+y^2)\circ c']\,(c')^*(dx) \wedge (c')^*(dy)$$

$$= (r^2\cos^2\theta + \ r^2\sin^2\theta)\ \left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) \wedge \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$

$$= r^2[(\cos\theta)dr - (r\sin\theta)d\theta] \wedge [(\sin\theta)dr + (r\cos\theta)d\theta]$$

$$= r^2[-r\sin^2(\theta)\,d\theta \wedge dr + (r\cos^2(\theta))dr \wedge d\theta$$

$$= r^2(r(\sin^2(\theta) + \cos^2(\theta))dr \wedge d\theta$$

$$= r^3 dr \wedge d\theta.$$

So we have the same integral as above:

$$\int_{c'} \omega = \int_{[0,1]\times[0,2\pi]} (c')^*\omega = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3\,dr\,d\theta = \frac{\pi}{2}\,.$$

Our new definition of $\int_c \ \omega$ is going to allow us to do line integrals and surface integrals (as well as integrals over higher dimensional regions).

Ex. Evaluate the line integral: $\int_c (x^5dx + y^2dy + zdz)$, where c is the curve $c(t) = (t, t^2, t^3), 0 \le t \le 1$ and $\omega = x^5dx + y^2dy + zdz$.

$$\int_c (x^5dx + y^2dy + zdz) = \int_0^1 c^*\omega.$$

$$c^*(x^5dx + y^2dy + zdz)$$

$$= (x^5 \circ c)c^*(dx) + (y^2 \circ c)c^*(dy) + (z \circ c)c^*(dz)$$

$$= t^5\left(\frac{\partial x}{\partial t}dt\right) + t^4\left(\frac{\partial y}{\partial t}dt\right) + t^3\left(\frac{\partial z}{\partial t}dt\right)$$

$$= t^5(dt) + t^4(2tdt) + t^3(3t^2dt) = 6t^5dt.$$

$$\int_c (x^5dx + y^2dy + zdz) = \int_0^1 6t^5dt = t^6|_0^1 = 1.$$

In a vector calculus course, that same integral might have been evaluated by:

$$c(t) = (t, t^2, t^3)$$

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2t \quad \frac{dz}{dt} = 3t^2$$

$$\int_c (x^5dx + y^2dy + zdz) = \int_0^1 t^5dt + t^4(2t)dt + t^3(3t^2)dt$$

$$= \int_0^1 6t^5dt = t^6|_0^1 = 1.$$

The formula that one learns in second year calculus for the integral of a 2-form over a surface in $\mathbb{R}^3$ is very messy. If a surface, S, is parameterized by $c(u,v) = \big(x(u,v), y(u,v), z(u,v)\big),\ \ (u,v) \in D$ and $\omega = Fdx \wedge dy + Gdy \wedge dz + Hdz \wedge dx$, then:

$$\iint_S \omega = \iint_S Fdx \wedge dy + Gdy \wedge dz + Hdz \wedge dx$$

$$= \iint_D \left[F\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial(x,y)}{\partial(u,v)} + G\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial(y,z)}{\partial(u,v)} + H\big(x(u,v), y(u,v), z(u,v)\big)\frac{\partial(z,x)}{\partial(u,v)}\right] dudv$$

where:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \text{ etc.}$$

This can be concisely summarized by:

$$\iint_S \omega = \iint_D c^*(\omega).$$

Ex. Find the surface integral:

$$\iint_S x\ dydz + y\ dxdy$$

where the surface S is given by:

$$\mathrm{c}(u,v) = (u+v, u^2 - v^2, uv)\ \ ;\ \ (u,v) \in [0,1]\ \times\ [0,1].$$

$$x = u + v \qquad y = u^2 - v^2 \qquad z = uv.$$

$$c^*(x\ dy \wedge dz + y\ dx \wedge dy)$$

$$= (x \circ c)c^*(dy) \wedge c^*(dz) + (y \circ c)c^*(dx) \wedge c^*(dy)$$

$$= (u+v)\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right) \wedge \left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right)$$

$$+(u^2 - v^2)\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) \wedge \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= (u+v)(2udu - 2vdv) \wedge (vdu + udv)$$

$$+ (u^2 - v^2)(du + dv) \wedge (2udu - 2vdv)$$

$$= (u+v)(2u^2 + 2v^2)du \wedge dv + (u^2 - v^2)(-2)(u+v)du \wedge dv$$

$$= 2(u+v)[u^2 + v^2 - (u^2 - v^2)]du \wedge dv = (4uv^2 + 4v^3)du \wedge dv.$$

$$\iint_S x\ dydz + y\ dxdy = \int_0^1 \int_0^1 (4uv^2 + 4v^3)du\ dv$$

$$= \int_0^1 2u^2v^2 + 4uv^3|_0^1\ dv = \int_0^1 2v^2 + 4v^3 dv$$

$$= \frac{2}{3}v^3 + v^4\Big|_0^1 = \frac{2}{3} + 1 = \frac{5}{3}.$$

The relationship between d and ∂ shows up in Stokes' Theorem.

Stokes' Theorem: If ω is a $(k-1)$-form on an open set, $A \subseteq \mathbb{R}^n$, and c is a k-chain in A, then:

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof: First let's prove this for $c = I^k$, so ω is a sum of $(k-1)$-forms of the type: $\quad f dx_1 \wedge \ldots \wedge \widehat{dx_\iota} \wedge \ldots \wedge dx_k$.

It is enough to show the theorem is true for each of these.

Recall that: $I^k_{(j,\alpha)}: I^k \to I^{k-1}$ by $I^k_{(j,\alpha)}(x) = (x_1, \ldots, \alpha, \ldots, x_k)$, where $\alpha = 0$ or 1 and is in the j^{th} place. So we have:

$$\left(I^k_{(j,\alpha)}\right)^* \left(f dx_1 \wedge \ldots \wedge \widehat{dx_\iota} \wedge \ldots \wedge dx_k\right)$$
$$= \left(f \circ I^k_{(j,\alpha)}\right)\left(I^k_{(j,\alpha)}\right)^*(dx_1) \wedge \ldots \wedge \widehat{\left(I^k_{(j,\alpha)}\right)^*(dx_\iota)} \wedge \ldots \wedge \left(I^k_{(j,\alpha)}\right)^*(dx_k)$$

$$= f(x_1, \ldots, \alpha, \ldots, x_k) dx_1 \wedge \ldots \wedge \widehat{dx_\iota} \wedge \ldots \wedge dx_k \qquad \text{if } j = l$$
$$= 0 \qquad \text{if } j \neq l.$$

This is because if $j \neq l$, then:

$$\left(I^k_{(j,\alpha)}\right)^*\left(dx_j\right) = \frac{\partial \alpha}{\partial x_j} dx_j = 0\ .$$

Thus we have what we will call Equation #1:

If $j \neq l$, then:

$$\int_{[0,1]^{k-1}} \left(I^k_{(j,\alpha)}\right)^* \left(f dx_1 \wedge \ldots \wedge \widehat{dx_\iota} \wedge \ldots \wedge dx_k\right) = 0$$

If $j = l$, then:

$$= \int_{[0,1]^k} f(x_1, \ldots, \alpha, \ldots\ x_k) dx_1 \ldots dx_k.$$

This is because:

$$\int_0^1 \int_0^1 f(x,1)dx\,dy = \int_0^1 \int_0^1 f(x,1)dy\,dx = \int_0^1 y\,f(x,1)\Big|_0^1 dx = \int_0^1 f(x,1)\,dx.$$

Now let's expand:

$$\int_{\partial I^k} f\,dx_1 \wedge \dots \wedge \widehat{dx_\iota} \wedge \dots \wedge dx_k$$
$$= \sum_{j=1}^{k}\sum_{\alpha=0}^{1}(-1)^{j+\alpha}\int_{I^k_{(j,\alpha)}} f dx_1 \wedge \dots \wedge \widehat{dx_\iota} \wedge \dots \wedge dx_k$$
$$= \sum_{j=1}^{k}\sum_{\alpha=0}^{1}(-1)^{j+\alpha}\int_{[0,1]^{k-1}} \left(I^k_{(j,\alpha)}\right)^* \left(f dx_1 \wedge \dots \wedge \widehat{dx_\iota} \wedge \dots \wedge dx_k\right)$$

By Equation #1 we have:

$$= (-1)^{l+1}\int_{[0,1]^k} f(x_1, \dots, 1, \dots, x_k)dx_1 \dots dx_k$$
$$+ (-1)^{l}\int_{[0,1]^k} f(x_1, \dots, 0, \dots, x_k)dx_1 \dots dx_k.$$

Now let's examine $\int_{I^k} d\left(f dx_1 \wedge \dots \wedge \widehat{dx_\iota} \wedge \dots \wedge dx_k\right)$:

$$\int_{I^k} d\left(f dx_1 \wedge \dots \wedge \widehat{dx_\iota} \wedge \dots \wedge dx_k\right) = \int_{I^k} \frac{\partial f}{\partial x_l} dx_l \wedge dx_1 \wedge \dots \wedge \widehat{dx_\iota} \wedge \dots \wedge dx_k$$
$$= (-1)^{l-1}\int_{I^k} \frac{\partial f}{\partial x_l} dx_1 \wedge \dots \wedge dx_l \wedge \dots \wedge dx_k$$
$$= (-1)^{l-1}\int_{I^k} \frac{\partial f}{\partial x_l} dx_1 \dots dx_k\ .$$

By Fubini's Theorem and the Fundamental Theorem of Calculus:

$$= (-1)^{l-1} \int_0^1 \dots \left(\int_0^1 \frac{\partial f}{\partial x_l} dx_l \right) dx_1 \dots \widehat{dx_l} \dots dx_k$$

$$= (-1)^{l-1} \int_0^1 \dots \int_0^1 [f(x, \dots, 1, \dots, x_k) - f(x_1, \dots, 0, \dots, x_k)] dx_1 \dots \widehat{dx_l} \dots dx_k$$

$$= (-1)^{l-1} \int_{[0,1]^k} f(x, \dots, 1, \dots, x_k) dx_1 \dots dx_k + (-1)^l \int_{[0,1]^k} f(x, \dots, 0, \dots, x_k) dx_1 \dots dx_k \, .$$

Thus we have:

$$\int_{I^k} d\left(f dx_1 \wedge \dots \wedge \widehat{dx_l} \wedge \dots \wedge dx_k\right) = \int_{\partial I^k} f dx_1 \wedge \dots \wedge \widehat{dx_l} \wedge \dots \wedge dx_k$$

So we can now write:

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega \, .$$

For a singular k-cube since

$$\int_{\partial c} \omega = \int_{\partial I^k} c^* \omega$$

we have:

$$\int_c d\omega = \int_{I^k} c^* d\omega = \int_{I^k} d(c^* \omega) = \int_{\partial I^k} c^* \omega = \int_{\partial c} \omega$$

Finally, if c is a k-chain, $\sum_{i=1}^{l} a_i c_i$, we have:

$$\int_c d\omega = \sum_{i=1}^{l} a_i \int_{c_i} d\omega = \sum_{i=1}^{l} a_i \int_{\partial c_i} \omega = \int_{\partial c} \omega.$$

Ex. Use Stokes' Theorem to show that $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is closed but not exact on $\mathbb{R}^2 - (0,0)$.

We already saw by direct calculation that $d\omega = 0$.
If $\omega = df$ on $\mathbb{R}^2 - (0,0)$, then by Stokes' Theorem if

$c(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$ we get:

$\int_c \omega = \int_c df = \int_{\partial c} f = 0;$ Since $\partial c = \phi$, but:

$$\int_c df = \int_{\partial c} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$c^*(\omega) = (-\sin t)(-\sin t)dt + (\cos t)(\cos t)\,dt = dt$$

$$\int_c \omega = \int_0^{2\pi} dt = 2\pi \ne 0$$

So ω is not exact on $\mathbb{R}^2 - (0,0)$.

Ex. Evaluate $\int_c \omega$ where c is the unit circle and:

$$\omega = ((\sin x)\,e^x - \cos y)dx + (x\sin y + \cos^3 y)dy.$$

By Stokes' Theorem:

$\int_c \omega = \int_D d\omega$ where $\partial D = c$ (i.e. D is the unit disk).

$d\omega = \sin y\, dy \wedge dx + \sin y\, dx \wedge dy = 0$, so we get:

$$\int_c \omega = \int_D 0 = 0.$$