Singular n-Chains

Def. A **singular $\boldsymbol{n}$-cube** in $A \subseteq \mathbb{R}^n$ is a continuous function:

$$c: [0,1]^n \to A$$

$$([0,1]^n = [0,1] \times [0,1] \times \ldots \times [0,1], n\text{-times})$$

Ex. $c: [0,1]^2 \to D \subseteq \mathbb{R}^2$, where D is the unit disk $x^2 + y^2 \leq 1$.

$$c(r,\theta) = (r\cos 2\pi\,\theta, r\sin 2\pi\,\theta).$$

Def. The **standard $\boldsymbol{n}$-cube** is the identity map on $[0,1]^n$:

$$I^n: [0,1]^n \to \mathbb{R}^n; \quad I^n(x) = x.$$

We will consider the formal sums of singular n-cubes of the form:
$3c_1 - 4c_2 + 5c_3$, where c_1, c_2, c_3 are singular n-cubes in A. A formal sum like this is called an **$\boldsymbol{n}$-chain** in A.

Def. Let $x \in [0,1]^{n-1}$. Define $I^n_{(i,0)}(x) = I^n(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$
and $I^n_{(i,1)}(x) = I^n(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})$.

$I^n_{(i,0)}$ is called the $\boldsymbol{(i,0)}$ **face** and $I^n_{(i,1)}$ is called the $\boldsymbol{(i,1)}$ **face**.

We define the boundary of I^n by:

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0}^{1} (-1)^{i+\alpha}\, I^n_{(i,\alpha)}\,.$$

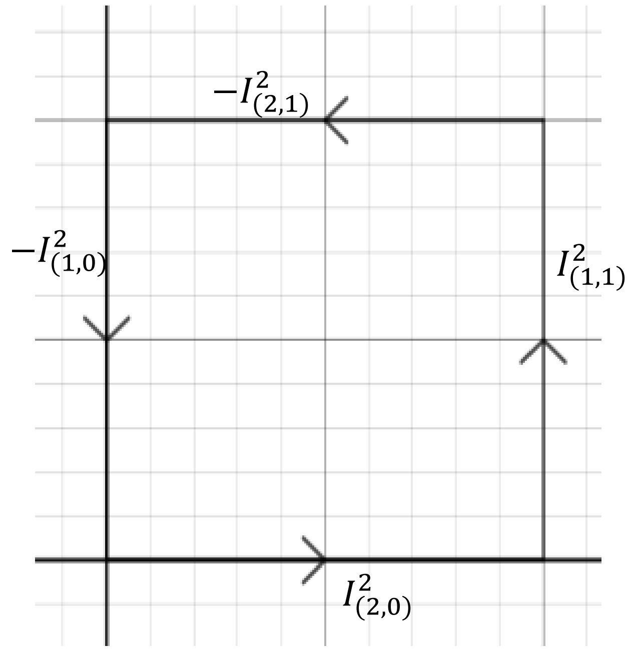
Ex. For $I^2 = [0,1]^2$:

$$I^2_{(1,0)} = I^2(0,x_1) = (0,x_1)$$

$$I^2_{(1,1)} = I^2(1,x_1) = (1,x_1)$$

$$I^2_{(2,0)} = I^2(x_1,0) = (x_1,0)$$

$$I^2_{(2,1)} = I^2(x_1,1) = (x_1,1)$$



$$\partial I^2 = \sum_{i=1}^{2}\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\, I^2_{(i,\alpha)} = -I^2_{(1,0)} + I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}$$

Def. For a general singular n-cube, $c: I^n \to A$, we define the (i,α) face as:

$$\boldsymbol{c_{(i,\alpha)} = c \circ I^n_{(i,\alpha)}}\,.$$

And we define the boundary of c by:

$$\boldsymbol{\partial c = \sum_{i=1}^{n}\sum_{\alpha=0}^{1}(-1)^{i+\alpha}\, c_{(i,\alpha)}}\,.$$

Ex. Let $c: [0,1]^2 \to \mathbb{R}^2$ by $c(r,\theta) = (r\cos(2\pi\,\theta), r\sin(2\pi\,\theta))$. Find ∂c.

We know from the previous example:

$$I^2_{(1,0)} = (0,\theta) \qquad \Rightarrow c_{(1,0)} = c(0,\theta) = (0,0)$$

$$I^2_{(1,1)} = (1,\theta) \qquad \Rightarrow c_{(1,1)} = c(1,\theta) = (\cos 2\pi\theta\,, \sin 2\pi\theta)$$

$$I^2_{(2,0)} = (r,0) \qquad \Rightarrow c_{(2,0)} = c(r,0) = (r,0)$$

$$I^2_{(2,1)} = (r,1) \qquad \Rightarrow c_{(2,1)} = c(r,1) = (r,0).$$

So we can write:

$$\partial c = \sum_{i=1}^{2} \sum_{\alpha=0}^{1} (-1)^{i+\alpha}\, c_{(i,\alpha)} = -c_{(1,0)} + c_{(1,1)} + c_{(2,0)} - c_{(2,1)}.$$

$c_{(2,0)} = c_{(2,1)}$ so $c_{(2,0)} - c_{(2,1)} = 0$, thus we know:

$$\partial c = -c_{(1,0)} + c_{(1,1)} = -(0,0) \cup (\cos 2\pi\theta\,, \sin 2\pi\theta).$$

Def. For an n-chain:

$$c = \sum_{i=1}^{j} a_i c_i\ .$$

We define $\boldsymbol{\partial c}$ by:

$$\boldsymbol{\partial c = \partial\left(\sum_{i=1}^{j} a_i c_i\right) = \sum_{i=1}^{j} a_i(\partial c_i)}.$$

Theorem: If c is an n-chain in A, then $\partial(\partial c) = 0$ or $\partial^2 = 0$.

This theorem is proved by showing that it's first true for I^k and then it follows for $c(I^k)$. We can see how this works for I^2.

We saw earlier that: $\partial I^2 = -I^2_{(1,0)} + I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}$, where:

$$I^2_{(1,0)} = (0, x_1)$$
$$I^2_{(1,1)} = (1, x_1)$$
$$I^2_{(2,0)} = (x_1, 0)$$
$$I^2_{(2,1)} = (x_1, 1)$$

$$\partial(\partial I^2) = \partial\left(-I^2_{(1,0)}\right) + \partial I^2_{(1,1)} + \partial I^2_{(2,0)} + \partial\left(-I^2_{(2,1)}\right)$$

$$= (0,0) - (0,1) + (1,1) - (1,0) + (1,0) - (0,0) + (0,1) - (1,1)$$

$$= 0.$$

We saw earlier for differential forms that:

$$d^2(\omega) = 0$$

Now we see that for n-chains:

$$\partial^2(c) = 0.$$

We also saw earlier that $d\omega = 0$ does not imply $\omega = d\eta$ (it depends on the geometry of the set $A \subseteq \mathbb{R}^n$). It's natural to ask for an n-chain if $\partial c = 0$, does it imply that $c = \partial k$ for some $(n-1)$-chain k?

The answer, generally, is no (and this is related to the statement about differential forms). For example, let $c: [0,1] \to \mathbb{R}^2 - (0,0)$ by:

$$c(t) = (\cos 2\pi t\,, \sin 2\pi t)$$

$$c(1) = c(0) = (1,0), \quad \text{so we know } \partial c = 0.$$

But there is no 2-chain k in $\mathbb{R}^2 - (0,0)$ with $\partial k = c$.