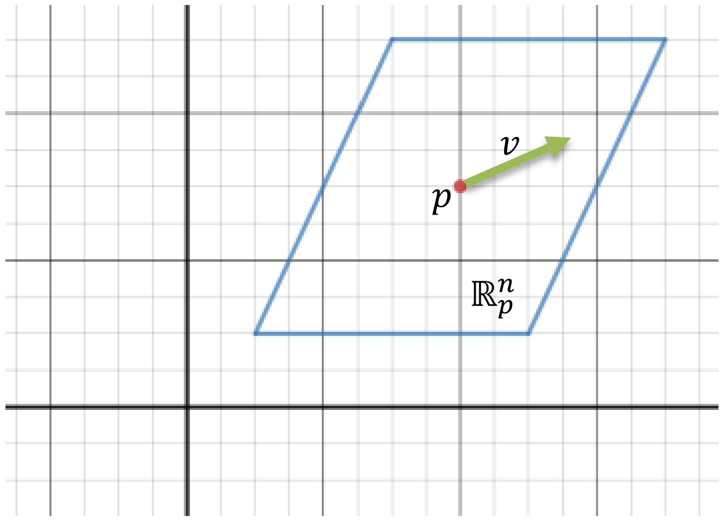
Vector Fields and Differential Forms on $\mathbb{R}^n$

If $p \in \mathbb{R}^n$, then the set of all pairs $(p, v), v \in \mathbb{R}^n$, is denoted $\mathbb{R}^n_p$, and called the **tangent space of $\mathbb{R}^n$ at p**.



Ex. The tangent plane to $(2, 5) \in \mathbb{R}^2$ is the set of all points $((2,5), v)$, $v \in \mathbb{R}^2$. Notice every tangent plane to the xy plane at any point looks like $\mathbb{R}^2$. To distinguish the $\mathbb{R}^2$ that is tangent to $(2, 5)$ from the $\mathbb{R}^2$ that is tangent to $(-1, -3)$ we define one by $\mathbb{R}^2_{(2,5)}$ and the other by $\mathbb{R}^2_{(-1,-3)}$.

Def. A **vector field** on $\mathbb{R}^n$ is a function, F, such that $F(p) \in \mathbb{R}^n_p$, for each $p \in \mathbb{R}^n$.

So if we let $(e_1)_p, (e_2)_p, \dots, (e_n)_p$ be the usual basis for $\mathbb{R}^n$ (i.e. we let $e_i = (0, 0, \dots, 1, 0, \dots, 0)$, with 1 in the i^{th} place), then we can write any vector field F as:

$$F(p) = F_1(p)(e_1)_p + \cdots + F_n(p)(e_n)_p$$

where $F_i: \mathbb{R}^n \to \mathbb{R}$. So the F_is are the components of the vector field. The vector field is continuous, differentiable, etc if the F_is are.

Given any two vector fields F, G and $f: \mathbb{R}^n \to \mathbb{R}$, we define:

$$(F+G)(p) = F(p) + G(p)$$
$$(F \cdot G)(p) = F(p) \cdot G(p)$$
$$(fF)(p) = \big(f(p)\big)\big(F(p)\big).$$

We define the **divergence of $\boldsymbol{F}$** by:

$$div(F) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

or we can write $\nabla \cdot F$, where:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right).$$

If $n = 3$ we define $\nabla \times F = \boldsymbol{curl(F)}$ as:

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

At each point $p \in \mathbb{R}^3$, $\nabla \times F$ is a vector in $\mathbb{R}^3_p$.

Suppose $\omega(p) \in \Omega^k(\mathbb{R}^n_p)$. Then, if $\varphi_1(p), \dots, \varphi_n(p)$ is the dual basis to $(e_1)_p, \dots, (e_n)_p$ (where $(e_1)_p, \dots, (e_n)_p$ is a basis for $\mathbb{R}^n_p$), i.e.

$$\big(\varphi_i(p)\big)\left(e_j\right)_p = \delta_{ij}.$$

Then we can write:

$$\omega(p) = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1}, \dots, {}_{i_k}(p)\left[\varphi_{i_1}(p) \wedge \dots \wedge \varphi_{i_k}(p)\right].$$

$\omega(p)$ is called a ***k*-form or differential form** on $\mathbb{R}^n$.

Thus a k-form, $\omega(p)$, is an alternating k-tensor on $\mathbb{R}^n_p$. So

$$\omega(p)\colon \mathbb{R}^n_p \times \dots \times \mathbb{R}^n_p \to \mathbb{R} \text{ is } k\text{-linear.}$$

A function, $f\colon \mathbb{R}^n \to \mathbb{R}$, is considered a **zero form**.

Recall that if $f\colon \mathbb{R}^n \to \mathbb{R}, f$ differentiable, then $Df(p)\colon \mathbb{R}^n \to \mathbb{R}$, where $Df(p)$ is a linear transformation and thus, $Df(p) \in \Omega^1(\mathbb{R}^n)$. We define a **1-form df** by: $\quad df(p)\left(v_p\right) = Df(p)(v).$

Let $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ by $\pi_i(x_1, \dots, x_n) = x_i$. Then we have:

$$dx_i(p)\left(v_p\right) = d\pi_i(p)\left(v_p\right) = D\pi_i(p)(v) = v_i$$

since the matrix representation of $D\pi_i$ is a 1 in the i^{th} place and zeroes everywhere else. That means:

$$d\pi_i(p)\left(e_j\right)_p = \delta_{ij}\ .$$

Thus $dx_1(p), \dots, dx_n(p)$ is the dual basis for $(e_1)_p, \ \dots, (e_n)_p$. Hence

$$dx_i(p)\left(e_j\right)_p = \delta_{ij}.$$

So we can write a k-form on $\mathbb{R}^n$ as:

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \dots, i_k} \ dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Ex. Suppose $\omega = xdx + y^2zdy - xzdz$. Find $\left(\omega(p)\right)(\vec{v})_p$ where $p = (1, -1, 2)$ and $\vec{v}_p = < 2, 3, 1 >_p$.

Since ω is a 1-form on $\mathbb{R}^3$, $\omega(p)$: $\mathbb{R}^3_p \to \mathbb{R}$ is a linear transformation.

$$\begin{aligned} \omega &= xdx + y^2zdy - xzdz \\ \omega(p) &= (1)dx + (-1)^2(2)dy - (1)(2)dz \\ &= dx + 2dy - 2dz \end{aligned}$$

$$\vec{v}_p = < 2, 3, 1 > = 2(e_1)_p + 3(e_2)_p + (e_3)_p \ .$$

So we have:

$$\left(\omega(p)\right)(\vec{v})_p = (dx + 2dy - 2dz)(2e_1 + 3e_2 + e_3)$$

$$= dx(2e_1 + 3e_2 + e_3) + 2dy(2e_1 + 3e_2 + e_3) - 2dz(2e_1 + 3e_2 + e_3)$$

Since $dx_i\left(e_j\right) = \delta_{ij}$ we get:

$$\left(\omega(p)\right)(\vec{v})_p = 2 + 2(3) - 2(1) = 6.$$

We know that if ω is a k-form and η is an l-form, then

$$\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega.$$

If ω is a k-form in $\mathbb{R}^3$, what can we say about $\omega \wedge \omega$?

If ω is a zero-form (i.e. just a real valued function $f: \mathbb{R}^3 \to \mathbb{R}$), then

$$\omega \wedge \omega = f^2(x).$$

If ω is a 1-form: $\quad \omega = F_1(x, y, z,)dx + F_2(x, y, z)dy + F_3(x, y, z)dz,$

then $\quad \omega \wedge \omega = (-1)^{(1)(1)}\omega \wedge \omega = -\omega \wedge \omega.$

But that implies $\omega \wedge \omega = 0$ (you can get the same result if you expand $\omega \wedge \omega$).

If ω is a 2-form (or a 3-form), then $\omega \wedge \omega = 0$. Why?

If we move to $\mathbb{R}^4$, then we can find a k-form, $k \neq 0$, where $\omega \wedge \omega \neq 0$.

Ex. If $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$

$$\omega \wedge \omega = (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$$

$$= dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_2 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$+ dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \wedge dx_3 \wedge dx_4$$

$$= 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable, then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

Proof:

$$df(p)(v_p) = Df(p)(v) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(p) & \dots & \frac{\partial f}{\partial x_n}(p) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(p) \right) v_i = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(p) \right) dx_i(p)(v_p)$$

So

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then $Df(p): \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. We can use this to define a linear transformation $f_*: \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$, defined by:

$$\boldsymbol{f_*(v_p) = \big(Df(p)(v)\big)_{f(p)}}.$$

Earlier we saw that if $g: V \to W$ is a linear transformation we could define another linear transformation, $g^*: \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ by:

$$g^*T(v_1, \dots, v_k) = T\big(g(v_1), \dots, g(v_k)\big)$$

where $T \in \mathcal{T}^k(W)$ and $v_1, \dots, v_k \in V$.

Similarly, we can define $f^*: \Omega^k\big(\mathbb{R}^m_{f(p)}\big) \to \Omega^k\big(\mathbb{R}^n_p\big)$ by:

$$\boldsymbol{f^*\omega(p)(v_1, \dots, v_k) = \omega\big(f(p)\big)\big(f_*(v_1), \dots, f_*(v_k)\big)}.$$

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then

1) $f^*(dx_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$

2) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

3) $f^*(g\omega) = (g \circ f)(f^*\omega)$, where $g: \mathbb{R}^n \to \mathbb{R}$.

4) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

5) If $h: \mathbb{R}^m \to \mathbb{R}^p$, then $(h \circ f)^*\omega = (f^* \circ h^*)\omega$

Proof of 1:

$$f^*(dx_i)(p)(v_p) = dx_i(f(p))\left(f_*(v_p)\right)$$

Now,

$$f_*(v_p) = \left(Df(p)(v)\right)_{f(p)}$$

$$f_*(v_p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \left(\sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} \; v_j \, , \ldots , \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j} \; v_j \right)$$

$$f^*(dx_i)(p)(v_p) = dx_i(f(p))\left(\sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j}\, v_j \,, \dots, \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j}\, v_j\right)$$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, v_j = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, dx_j(p)(v_p)$$

So:

$$f^*(dx_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, dx_j \,.$$

Ex. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x,y,z) = (xyz, yz, yz^2)$.
Let $\omega = (s+rt)dr \wedge dt$ be a 2-form on the second $\mathbb{R}^3$; (r,s,t).
Find $f^*(\omega)$.

$$f^*\big((s+rt)dr \wedge dt\big) = \big((s+rt)\circ f\big)f^*(dr \wedge dt) \qquad \text{by \#3}$$

$$= (yz + xy^2z^3)f^*(dr \wedge dt)$$

$$= (yz + xy^2z^3)\big(f^*(dr) \wedge f^*(dt)\big) \qquad \text{by \#4}$$

$$= (yz + xy^2z^3)\Big[\left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz\right) \wedge \left(\frac{\partial f_3}{\partial x}dx + \frac{\partial f_3}{\partial y}dy + \frac{\partial f_3}{\partial z}dz\right)\Big]$$

$$= (yz + xy^2z^3)[(yz\,dx + xz\,dy + xy\,dz) \wedge (z^2dy + 2yz\,dz)]$$

$$= (yz + xy^2z^3)[yz^3 dx \wedge dy + 2y^2z^2 dx \wedge dz + xz^3 dy \wedge dy$$
$$+ 2xyz^2 dy \wedge dz + xyz^2 dz \wedge dy + 2xy^2z\, dz \wedge dz]$$

$dy \wedge dy = dz \wedge dz = 0$ since $dy \wedge dy = -dy \wedge dy$, etc
and $dz \wedge dy = -dy \wedge dz$; so we can write:

$$= (yz + xy^2z^3)(yz^3 dx \wedge dy + 2y^2z^2 dx \wedge dz + (2xyz^2 - xyz^2) dy \wedge dz)$$

$$= (yz + xy^2z^3)(yz^2 dx \wedge dy + 2y^2z^2 dx \wedge dz + xyz^2 dy \wedge dz).$$

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, then
$$f^*(h\, dx_1 \wedge \ldots \wedge dx_n) = (h \circ f)(\det f') dx_1 \wedge \ldots \wedge dx_n .$$

Let $\omega \in \Omega^k(\mathbb{R}^n_p)$, a k-form on $\mathbb{R}^n$.

We define a map $d: \Omega^k(\mathbb{R}^n_p) \to \Omega^{k+1}(\mathbb{R}^n_p)$, called the differential of ω by:

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

$$d\omega = \sum_{i_1 < \cdots < i_k} (d\omega_{i_1,\ldots,i_k}) \wedge dx_i \wedge \ldots \wedge dx_{i_k}$$

where

$$d\omega_{i_1,\ldots,i_k} = \sum_{j=1}^{n} \frac{\partial \omega_{i_1,\ldots,i_k}}{\partial x_j} dx_j.$$

Theorem:

1) $d(\omega+\eta)=d\omega+d\eta$

2) If ω is a k-form and η is an l-form, then:
$$d(\omega\wedge\eta)=d\omega\wedge\eta+(-1)^k\omega\wedge d\eta$$

3) $d(d\omega)=0$

4) If ω is a k-form on $\mathbb{R}^m$ and $f:\mathbb{R}^n\to\mathbb{R}^m$ is differentiable, then:
$$f^*(d\omega)=d(f^*\omega).$$

Proofs of 1-3:

1) Let: $\omega=\sum_{1\le i_1<\cdots<i_k\le n}\omega_{i_1,\dots,i_k}dx_{i_1}\wedge\ldots\wedge dx_{i_k}$

$$\eta=\sum_{1\le i_1<\cdots<i_k\le n}\eta_{i_1,\dots,i_k}dx_{i_1}\wedge\ldots\wedge dx_{i_k}$$

Note: some coefficients may be 0

Then:

$$d(\omega+\eta)=d\left(\sum_{1\le i_1<\cdots<i_k\le n}\left(\omega_{i_1,\dots,i_k}+\eta_{i_1,\dots,i_k}\right)dx_{i_1}\wedge\ldots\wedge dx_{i_k}\right)$$

$$=\sum_{1\le i_1<\cdots<i_k\le n}d\left(\omega_{i_1,\dots,i_k}+\eta_{i_1,\dots,i_k}\right)\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}$$

$$=\sum_{1\le i_1<\cdots<i_k\le n}\left[d\left(\omega_{i_1,\dots,i_k}\right)+d\left(\eta_{i_1,\dots,i_k}\right)\right]\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}$$

$$=d\omega+d\eta\,.$$

2) By number 1, it's enough to show 2 is true for:

$$\omega = g dx_{i_1} \wedge \ldots \wedge dx_{i_k} \text{ and } \eta = h dx_{j_1} \wedge \ldots \wedge dx_{j_l}.$$

$d(\omega \wedge \eta) = d\left(gh\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l}\right)$.

Notice that:

$$d(gh) = \sum_{\alpha=1}^{n} \frac{\partial(gh)}{\partial x_\alpha} dx_\alpha = \sum_{\alpha=1}^{n} \left(g \frac{\partial h}{\partial x_\alpha} + h \frac{\partial g}{\partial x_\alpha}\right) dx_\alpha.$$

$$\begin{aligned} d(\omega \wedge \eta) &= \textstyle\sum_{\alpha=1}^{n} \left(g \frac{\partial h}{\partial x_\alpha} + h \frac{\partial g}{\partial x_\alpha}\right) dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \\ &= \sum_{\alpha=1}^{n} \frac{\partial h}{\partial x_\alpha}\left(g dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l}\right) \\ &\quad + \sum_{\alpha=1}^{n} \frac{\partial g}{\partial x_\alpha}\left(dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge h dx_{j_1} \wedge \ldots \wedge dx_{j_l}\right). \end{aligned}$$

The second term is just $d\omega \wedge \eta$. But in the first term, notice:

$$\begin{aligned} &\sum_{\alpha=1}^{n} \frac{\partial h}{\partial x_\alpha} g dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \\ &= (-1)^k \sum_{\alpha=1}^{n} g dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge \left(\frac{\partial h}{\partial x_\alpha}\right) dx_\alpha \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \\ &= (-1)^k \omega \wedge d\eta. \end{aligned}$$

So $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

3) The fact that $d(d\omega) = 0$ follows from: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

Let's show $d(d\omega) = 0$ for $\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

$$d\omega = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha} dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

$$d(d\omega) = \sum_{\beta=1}^{n} \sum_{\alpha=1}^{n} \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} dx_\beta \wedge dx_\alpha \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

$\frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}$ and $\frac{\partial^2 f}{\partial x_\beta \partial x_\alpha}$ appear in pairs and $dx_\beta \wedge dx_\alpha = -dx_\alpha \wedge dx_\beta$.

So $d(d\omega) = 0$.

Ex. Let $\omega = xyzdx + y^2 zdz$ be a 1-form on $\mathbb{R}^3$. Find $d\omega$.

$$d\omega = d(xyzdx + y^2 zdz) = d(xyzdx) + d(y^2 zdz)$$

$$= d(xyz) \wedge dx + d(y^2 z) \wedge dz$$

$$= \left(\frac{\partial(xyz)}{\partial x} dx + \frac{\partial(xyz)}{\partial y} dy + \frac{\partial(xyz)}{\partial z} dz\right) \wedge dx$$

$$+ \left(\frac{\partial(y^2 z)}{\partial x} dx + \frac{\partial(y^2 z)}{\partial y} dy + \frac{\partial(y^2 z)}{\partial z} dz\right) \wedge dz$$

$$= (yzdx + xzdy + xydz) \wedge dx + (2yzdy + y^2 dz) \wedge dz$$

$$= yzdx \wedge dx + xzdy \wedge dx + xydz \wedge dx + 2yzdy \wedge dz + y^2 dz \wedge dz$$

$$= -xzdx \wedge dy - xydx \wedge dz + 2yzdy \wedge dz.$$

Ex. Let $\omega = ze^x dx \wedge dy + xy^2 dx \wedge dz$ be a 2-form on $\mathbb{R}^3$. Find $d\omega$.

$$d\omega = d(ze^x) \wedge dx \wedge dy + d(xy^2) \wedge dx \wedge dz$$

$$= e^x dz \wedge dx \wedge dy + 2xy\, dy \wedge dx \wedge dz$$

$$= e^x dx \wedge dy \wedge dz - 2xy\, dx \wedge dy \wedge dz$$

$$= (e^x - 2xy) dx \wedge dy \wedge dz.$$