Vector Fields and Differential Forms on $\mathbb{R}^n$- HW Problems

1. Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x, y, z) = (ye^x, z^2, xz)$.
   a. Let $\omega = (xy)dy \wedge dz$. Find $f^*\omega$.
   b. Let $\eta = xzdx + dy + zdz$. Find $f^*\eta$.
   c. Find $f^*(\eta \wedge \omega)$.

2. Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (xy, yz)$, i.e. $u = xy, \quad v = yz$. Let $\omega = (u^2v)du \wedge dv$. Find $f^*\omega$.

3. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(r, \theta) = (rcos\theta, rsin\theta)$, i.e. $x = rcos\theta,\ y = rsin\theta$. Let $\omega = dx \wedge dy$. Find $f^*\omega$.

4. Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ by $f(r, \theta) = (rcos2\pi\theta,\ rsin2\pi\theta,\ \sqrt{1 - r^2})$. Find $f^*(xdx \wedge dy + zdz \wedge dx)$.

5. Let $\omega = ydx + (xz)dy$ and $\eta = (xy)dy - xdz$ be 1-forms on $\mathbb{R}^3$.
   a. Calculate $\omega \wedge \eta$.
   b. Find $d(\omega \wedge \eta)$.
   c. Calculate $d\omega$ and $d\eta$ and show that
   $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\ \omega \wedge d\eta.$$

6. Let $f: \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = xye^z$.
   a. Calculate df.
   b. Calculate $d(df)$.

7. Let $\omega = x_2x_3dx_1 - 2x_1dx_2 + x_4dx_3 - x_2dx_4$ be a 1-form on $\mathbb{R}^4$. Let p be the point $p = (1, -2, -1, 2) \in \mathbb{R}^4$ and $\vec{v} =< 2, 3, -2, 1 >$ be a vector in $\mathbb{R}^4_p$. Find $(\omega(p))(\vec{v})$.

8. Does there exist a 3-form in $\mathbb{R}^6$ such that $\omega \wedge \omega \neq 0$? If so, find one If not, prove it can't exist.