Calculations with the Metric Tensor

Now let's apply these tensor concepts to the metric tensor for a surface in $\mathbb{R}^3$. Recall that if a surface, S, in $\mathbb{R}^3$ is parameterized by:

$$\vec{\Phi}: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$$
$$\vec{\Phi}(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$$

then, the first fundamental form, or metric tensor, is given by:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where

$$g_{11} = \vec{\Phi}_u \cdot \vec{\Phi}_u$$
$$g_{12} = g_{21} = \vec{\Phi}_u \cdot \vec{\Phi}_v$$
$$g_{22} = \vec{\Phi}_v \cdot \vec{\Phi}_v \,.$$

At each point, $p \in S$, $\;g \in \mathcal{T}^2\big(T_pS\big)$.

Thus if given $\vec{w}_1, \vec{w}_2 \in T_pS$, then:

$$g(\vec{w}_1, \vec{w}_2) = \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$$

where:

$$\vec{w}_1 = a_{11}\vec{\Phi}_u + a_{12}\vec{\Phi}_v$$
$$\vec{w}_2 = a_{21}\vec{\Phi}_u + a_{22}\vec{\Phi}_v \,.$$

Ex. Let S be the surface parameterized by $\vec{\Phi}: \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ and

$\vec{\Phi}(u,v) = (u, v, u^2 + v^2)$.

a) Find the metric tensor, g, at $(u,v) = (1,2)$.

b) If $\vec{w}_1 = 2\vec{\Phi}_u(1,2) - 3\vec{\Phi}_v(1,2)$ and $\vec{w}_2 = -\vec{\Phi}_u(1,2) + 2\vec{\Phi}_v(1,2)$

then find $g\left(\vec{\Phi}_u(1,2), \vec{\Phi}_v(1,2)\right)$, $g(\vec{w}_1, \vec{w}_2)$.

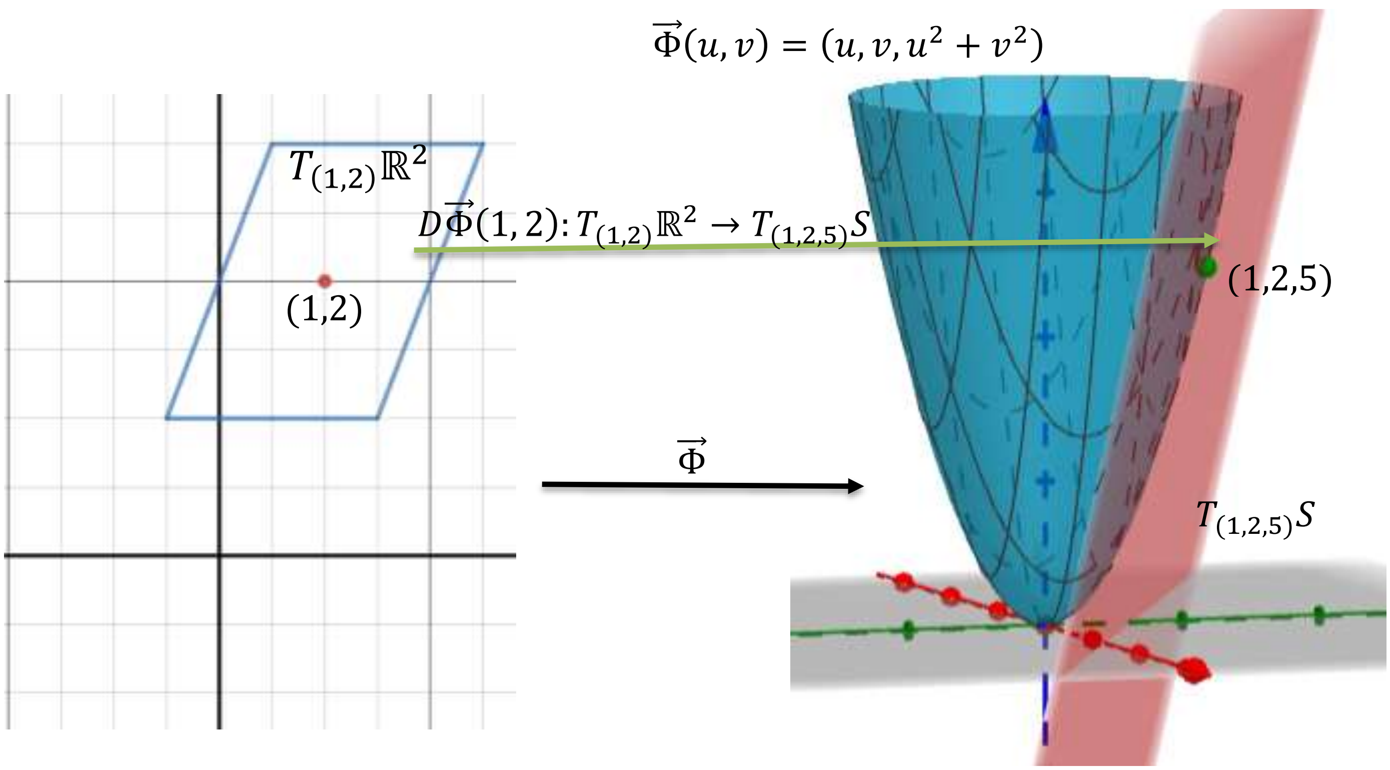
c) We know that $D\vec{\Phi}(1,2): T_{(1,2)}\mathbb{R}^2 \to T_{(1,2,5)}S$ is a linear transformation.

If $U \in \mathcal{T}^k\left(T_{(1,2,5)}S\right)$, then

$\left(D\vec{\Phi}(1,2)\right)^* U \in \mathcal{T}^k\left(T_{(1,2)}\mathbb{R}^2\right)$.

Find $\left(D\vec{\Phi}(1,2)\right)^* g(\vec{v}_1, \vec{v}_2)$ where $\vec{v}_1 = (-3,2)$, $\vec{v}_2 = (1,-1)$

and $\vec{v}_1, \vec{v}_2 \in T_{(1,2)}\mathbb{R}^2$.



a) $\vec{\Phi}(u,v) = (u, v, u^2 + v^2)$

$$\vec{\Phi}_u = (1,0,2u) \qquad \vec{\Phi}_u(1,2) = (1,0,2)$$
$$\vec{\Phi}_v = (0,1,2v) \qquad \vec{\Phi}_v(1,2) = (0,1,4)$$

$$g_{11} = \vec{\Phi}_u \cdot \vec{\Phi}_u = 5$$
$$g_{12} = g_{21} = \vec{\Phi}_u \cdot \vec{\Phi}_v = 8$$
$$g_{22} = \vec{\Phi}_v \cdot \vec{\Phi}_v = 17$$

So $g = \begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}$ is the metric tensor at $(u,v) = (1,2)$.

b) $\vec{\Phi}_u = (1,0), \vec{\Phi}_v = (0,1)$ since Φ_u and Φ_v are the basis vectors for $T_{(1,2,5)}S$, so:

$$g(\vec{\Phi}_u, \vec{\Phi}_v) = (1 \quad 0)\begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= (1 \quad 0)\begin{pmatrix} 8 \\ 17 \end{pmatrix} = 8.$$

$$\vec{w}_1 = (2,-3); \qquad \vec{w}_2 = (-1,2)$$

$$g(\vec{w}_1, \vec{w}_2) = (2 \quad -3)\begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= (2 \quad -3)\begin{pmatrix} 11 \\ 26 \end{pmatrix} = -56.$$

c) $D\vec{\Phi}(1,2) = \left(\vec{\Phi}_u(1,2) \quad \vec{\Phi}_v(1,2)\right)$

$$\vec{v}_1 = (-3,2); \qquad \vec{v}_2 = (1,-1).$$

$$\left(D\vec{\Phi}(1,2)\right)^* g(\vec{v}_1, \vec{v}_2) = g\left(D\vec{\Phi}(1,2)\vec{v}_1, D\vec{\Phi}(1,2)\vec{v}_2\right)$$

$$\left(D\vec{\Phi}(1,2)\right)(-3,2) = \left(\vec{\Phi}_u(1,2) \quad \vec{\Phi}_v(1,2)\right)(-3,2)$$
$$= -3\vec{\Phi}_u + 2\vec{\Phi}_v = (-3,2)$$

$$\left(D\vec{\Phi}(1,2)\right)(1,-1) = \left(\vec{\Phi}_u(1,2) \quad \vec{\Phi}_v(1,2)\right)(1,-1)$$
$$= \vec{\Phi}_u - \vec{\Phi}_v = (1,-1).$$

Notice that:

$$D\vec{\Phi}(1,2\,) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}.$$

So we can write:

$$\left(D\vec{\Phi}(1,2\,)\right)(-3 \quad 2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}.$$

But in the basis $\vec{\Phi}_u(1,2) = (1,0,2)$ and $\vec{\Phi}_v(1,2) = (0,1,4)$:

$$(-3,2,2) = -3\vec{\Phi}_u + 2\vec{\Phi}_v\,.$$

Thus,

$$\left(D\vec{\Phi}(1,2)\right)^* g(\vec{v}_1, \vec{v}_2) = g\left((-3,2),(1,-1)\right)$$
$$= (-3, \quad 2)\begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= (-3 \quad 2)\begin{pmatrix} -3 \\ -9 \end{pmatrix} = -9.$$

We had a theorem that said if $v_1, \ldots, v_n$ is a basis for V and $\varphi_1, \ldots, \varphi_n$ is a dual basis for V^*, i.e. $\varphi_i(v_j) = \delta_{ij}$, then we can write any m tensor as a linear combination of $\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_m}$ where $1 \le i_1, \ldots, i_m \le n$.

So how do we write the metric tensor as a linear combination of 2-tensors of the form $\varphi_i \otimes \varphi_j$ where $\{\varphi_1, \varphi_2\}$ is the dual basis for $(T_pS)^*$?
In this case, $V = T_pS$ and the basis vectors for V are $\overrightarrow{\Phi}_u$ and $\overrightarrow{\Phi}_v$. The dual basis for φ_1 and φ_2 have the properties that:

$$\varphi_1(\overrightarrow{\Phi}_u) = 1 \qquad \varphi_2(\overrightarrow{\Phi}_u) = 0$$
$$\varphi_1(\overrightarrow{\Phi}_v) = 0 \qquad \varphi_2(\overrightarrow{\Phi}_v) = 1.$$

Note: φ_1 is generally written as du and φ_2 is generally written as dv (i.e. φ_1 and φ_2 are differential forms).

So we would like to be able to write:

$$g = \lambda_1(\varphi_1 \otimes \varphi_1) + \lambda_2(\varphi_1 \otimes \varphi_2) + \lambda_3(\varphi_2 \otimes \varphi_1) + \lambda_4(\varphi_2 \otimes \varphi_2).$$

What are $\lambda_1, \lambda_2, \lambda_3$, and λ_4?

If $\vec{v}_1 = a_1\overrightarrow{\Phi}_u + b_1\overrightarrow{\Phi}_v$ and $\vec{v}_2 = a_2\overrightarrow{\Phi}_u + b_2\overrightarrow{\Phi}_v$, then:

$$g\big((a_1, b_1), (a_2, b_2)\big) = (a_1 \quad b_1)\begin{pmatrix} 5 & 8 \\ 8 & 17 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$
$$= 5a_1a_2 + 8a_1b_2 + 8a_2b_1 + 17b_1b_2.$$

Now let's expand the RHS of the tensor equation:

$$\big(\lambda_1(\varphi_1 \otimes \varphi_1) + \lambda_2(\varphi_1 \otimes \varphi_2) + \lambda_3(\varphi_2 \otimes \varphi_1) + \lambda_4(\varphi_2 \otimes \varphi_2)\big)\big((a_1, b_1), (a_2, b_2)\big)$$

$$= \lambda_1 \varphi_1(a_1, b_1)\varphi_1(a_2, b_2) + \lambda_2 \varphi_1(a_1, b_1)\varphi_2(a_2, b_2) + \lambda_3 \varphi_2(a_1, b_1)\varphi_1(a_2, b_2) + \lambda_4 \varphi_2(a_1, b_1)\varphi_2(a_2, b_2)$$

$$= \lambda_1(a_1)(a_2) + \lambda_2(a_1)(b_2) + \lambda_3(a_2)(b_1) + \lambda_4(a_2)(b_2).$$

Thus since

$g\big((a_1, b_1), (a_2, b_2)\big) = 5a_1a_2 + 8a_1b_2 + 8a_2b_1 + 17b_1b_2$

We have:

$\lambda_1 = 5,\ \lambda_2 = 8,\ \lambda_3 = 8,\ \lambda_4 = 17$ and

$$g = 5\varphi_1 \otimes \varphi_1 + 8\varphi_1 \otimes \varphi_2 + 8\varphi_2 \otimes \varphi_1 + 17\varphi_2 \otimes \varphi_2$$

or

$$g = 5du \otimes du + 8du \otimes dv + 8dv \otimes du + 17dv \otimes dv.$$