Functions on $\mathbb{R}^n$

The Topology of $\mathbb{R}^n$

$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) |\ x_i \in \mathbb{R},\ i = 1, \dots, n\}.$

$\mathbb{R}^n$ is a vector space with standard basis $\{e_1, e_2, \dots, e_n\}$ where:

$$
\begin{aligned}
e_1 &= (1, 0, 0, \dots ,0)\\
e_2 &= (0, 1, 0, \dots ,0)\\
&\vdots\\
e_n &= (0, 0, 0, \dots ,1).
\end{aligned}
$$

We define a norm on $\mathbb{R}^n$ by:

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\ ; \quad x = (x_1, x_2, \dots, x_n).$$

We can then define a distance on $\mathbb{R}^n$ by:

$$d(x, y) = |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

where $\quad x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n).$

Def. Let $\{p_j\}$ be a sequence in $\mathbb{R}^n$. We say $\{\boldsymbol{p_j}\}$ **converges to** $\boldsymbol{p} \in \mathbb{R}^n$ if for all $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ (ie, the positive integers) such that if $j \geq N$ then $|p - p_j| < \epsilon$.

Def. Let $\{p_j\}$ be a sequence in $\mathbb{R}^n$. We say $\{\boldsymbol{p_j}\}$ is a **Cauchy sequence** if for all $\epsilon > 0$ there exists a $N \in \mathbb{Z}^+$ such that if $j, k \geq N$ then $|p_j - p_k| < \epsilon$.

$\mathbb{R}^n$ is **complete** (i.e. every Cauchy sequence converges) with respect to this distance function. In addition, $\mathbb{R}^n$ is a **Banach space** (i.e. a complete, normed, vector space).

Proposition: Given $x, y \in \mathbb{R}^n$, then:

i) $|x + y| \le |x| + |y|$ (triangle inequality)

ii) $|x \cdot y| \le |x|\,|y|$ (Cauchy-Schwarz inequality).

Def. A **linear transformation**, $T: \mathbb{R}^n \to \mathbb{R}^m$, is a function such that for all $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

a. $T(u + v) = T(u) + T(v)$

b. $T(cu) = cT(u)$.

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ can be represented with respect to the usual basis in $\mathbb{R}^n$ and $\mathbb{R}^m$ by an $m \times n$ matrix

$$T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where $T(e_i) = \sum_{j=1}^{m} a_{ji} e_j$, $e_j = (0, 0, \dots, 1, 0, 0, \dots, 0)$ and the 1 is in the j^{th} place.

The coefficients of $T(e_i)$ appear in the i^{th} column of the matrix.

$$T(e_i) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}.$$

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^4$ and $S: \mathbb{R}^4 \to \mathbb{R}^3$ be linear transformations. Suppose:

$$T(1,0) = (0,2,3,1) \qquad S(1,0,0,0) = (1,2,3)$$
$$T(0,1) = (2,-1,-1,2) \qquad S(0,1,0,0) = (-1,3,1)$$
$$S(0,0,1,0) = (2,3,1)$$
$$S(0,0,0,1) = (0,1,2).$$

Find a matrix representation of S and T with respect to the standard basis, then find a matrix representation of $S \circ T: \mathbb{R}^2 \to \mathbb{R}^3$.

$$T = \begin{pmatrix} 0 & 2 \\ 2 & -1 \\ 3 & -1 \\ 1 & 2 \end{pmatrix}; \qquad S = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 3 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}.$$

The matrix representation of the composition, $S \circ T$, is gotten by matrix multiplication.

$$S \circ T = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 3 & 3 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & -1 \\ 3 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 16 & 0 \\ 7 & 8 \end{pmatrix}.$$

Prop. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a number, M, such that: $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^n$.

Proof: Let $h = (h_1, h_2, \ldots, h_n)$ and $T = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$ then,

$$|T(h)| = \left| \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \right|$$

$$= \begin{vmatrix} a_1 \cdot h \\ a_2 \cdot h \\ \vdots \\ a_m \cdot h \end{vmatrix} \qquad \text{where } a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$$

$$= \sqrt{(a_1 \cdot h)^2 + (a_2 \cdot h)^2 + \cdots + (a_m \cdot h)^2}$$

$$\leq \sqrt{(|a_1||h|)^2 + (|a_2||h|)^2 + \cdots + (|a_m||h|)^2} \quad \text{Cauchy-Schwarz Inequality}$$

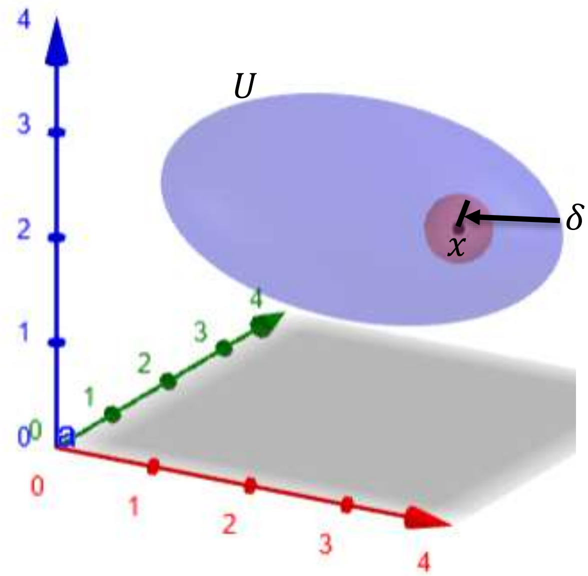
$$= \left(\sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_m|^2}\right)|h|.$$

Thus:

$$|T(h)| \leq \left(\sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_m|^2}\right)|h|.$$

So take $$M = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_m|^2}.$$

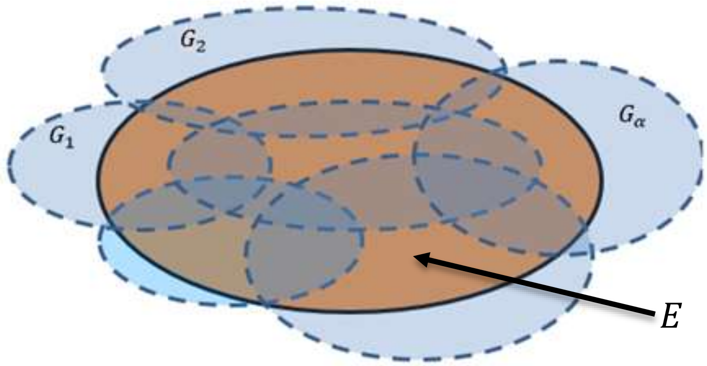
Def. A subset $U \subseteq \mathbb{R}^n$ is **open** if given any point $x \in U$, x is an **interior point** of U. That is, there exists a $\delta > 0$ such that if
$$d(x,y) = |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} < \delta,$$
then $y \in U$.



Ex. $H = \{(x_1, \ldots, x_n) |\ x_n > 0\}$ is an open set in $\mathbb{R}^n$. Given any point, $(x_1, \ldots, x_n) \in H$, the set of $y \in \mathbb{R}^n$ where:
$$\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} < \frac{x_n}{2} = \delta$$
is contained in H.

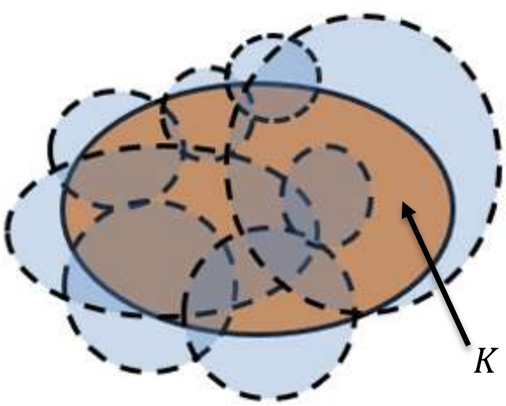
Def. A subset $V \subseteq \mathbb{R}^n$ is **closed** if its complement in $\mathbb{R}^n$, i.e. $\mathbb{R}^n - V$, is open .

Def. Let $E \subseteq \mathbb{R}^n$. $\{G_\alpha\}$ **is an open cover of E** if each G_α is an open set and $\bigcup_\alpha G_\alpha \supseteq E$.

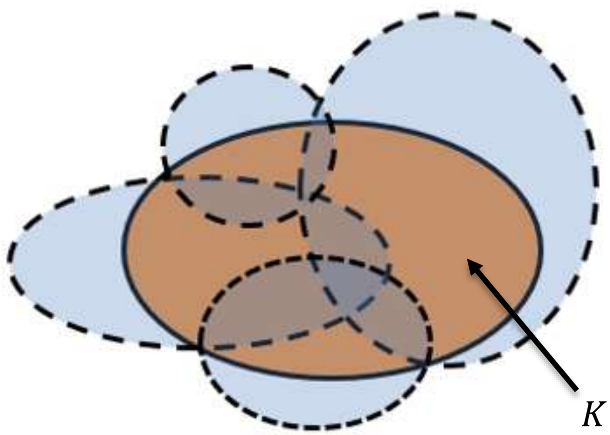


Open Cover of E

Def. A set K is **compact** if every open cover has a finite subcover.



Open Cover of K



Finite Subcover of K

Theorem (Heine-Borel): If $K \subseteq \mathbb{R}^n$, K is compact if, and only if, K is closed and bounded.

Ex. $[0, 3]$ is compact.

$[0,1] \times [0,1] \times [0,1]$ is compact.

$(0, 3]$ is not compact (not closed).

Functions on $\mathbb{R}^n$

Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then we can write:

$$f(x_1, x_2, \dots, x_n) = \big(f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_m(x_1, x_2, \dots, x_n)\big)$$

where $f_i: A \subseteq \mathbb{R}^n \to \mathbb{R}$.

Def. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $x, a \in \mathbb{R}^n, p \in \mathbb{R}^m$. We say $\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - p| < \epsilon$.

Notice, if $x = (x_1, \dots, x_n), a = (a_1, \dots, a_n)$, and $p = (p_1, \dots, p_m)$, then the δ and ϵ statements become:

$$0 < \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} < \delta \quad \text{implies that}$$

$$\sqrt{(f_1(x) - p_1)^2 + \cdots + (f_m(x) - p_m)^2} < \epsilon .$$

The following propositions will be useful later:

Prop: Let $f: \mathbb{R}^n \to \mathbb{R}^m$, then $\lim_{x \to a} f(x) = \vec{0}$ (i.e. the zero-vector in $\mathbb{R}^m$) if, and only if, $\lim_{x \to a} |f(x)| = 0$.

Proof: Assume $\lim_{x \to a} f(x) = \vec{0}$.

We must show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $\big||f(x)| - 0\big| < \epsilon$.

Notice: $\big||f(x)| - 0\big| = \big|f(x) - \vec{0}\big|$.

But since $\lim_{x \to a} f(x) = \vec{0}$, we know given any $\epsilon > 0$, there exists a $\delta' > 0$ such that if $0 < |x - a| < \delta'$, then $\big|f(x) - \vec{0}\big| < \epsilon$.

Choose $\delta = \delta'$ then $0 < |x - a| < \delta$ implies

$$\big||f(x)| - 0\big| = \big|f(x) - \vec{0}\big| < \epsilon.$$

$$\Rightarrow \lim_{x \to a} |f(x)| = 0.$$

Now assume $\lim_{x \to a} |f(x)| = 0$.

We must show given any $\epsilon > 0$ there exists a $\delta > 0$ such that if

$0 < |x - a| < \delta$, then $\big|f(x) - \vec{0}\big| < \epsilon$.

But since $\lim_{x \to a} |f(x)| = 0$, we know given $\epsilon > 0$ there exists a $\delta' > 0$ such that if

$0 < |x - a| < \delta'$, then $\big||f(x)| - 0\big| < \epsilon$.

Choose $\delta = \delta'$ then if $0 < |x - a| < \delta$, then

$$\big|f(x) - \vec{0}\big| = \big||f(x)| - 0\big| < \epsilon.$$

$$\Rightarrow \quad \lim_{x \to a} f(x) = \vec{0}.$$

Prop: Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $a, x \in \mathbb{R}^n$, then $\lim\limits_{x\to a}|f(x)| = 0$ if,

and only if, $\lim\limits_{x\to a}|f_i(x)| = 0$ for $i = 1, \dots, m$, and

$f(x) = (f_1(x), f_2(x), \dots, f_m(x))$.

Proof: HW Problem: For one direction of this proof it is useful to know:

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \le \sqrt{x_1^2} + \sqrt{x_2^2} + \cdots \sqrt{x_n^2}$$
$$= |x_1| + |x_2| + \cdots |x_n|.$$

We can see this by squaring both sides of the inequality:

$$x_1^2 + x_2^2 + \cdots + x_n^2 \le |x_1|^2 + \cdots |x_n|^2 + \textstyle\sum_{i<j} 2|x_i||x_j|.$$

Def. f is **continuous** at $x = (x_1, x_2, \dots, x_n) \in A$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such that if $d(x, y) < \delta$, $y \in A$, then $d\big(f(x), f(y)\big) < \epsilon$.

That is, if $\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} < \delta$

Then $\sqrt{\big(f_1(x) - f_1(y)\big)^2 + \cdots + \big(f_m(x) - f_m(y)\big)^2} < \epsilon$.

Def. f is **continuous on A** $\subseteq \mathbb{R}^n$ if f is continuous at every point $x \in A$.

Theorem: $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous if, and only if, each f_i is continuous.

Proof: HW Problem.

Theorem: Let $f, g: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be continuous at $a \in U$, then
$f \pm g, |f|$, and $f \cdot g$ are continuous at $a \in U$.

Proof of $f + g$ is continuous at $a \in U$.

We must show for all $\epsilon > 0$ there exists an $\delta > 0$ such that if
$|x - a| < \delta$, then $\left|\left(f(x) + g(x)\right) - \left(f(a) + g(a)\right)\right| < \epsilon$.

Since f is continuous at $a \in U$, we know there exists a $\delta_1 > 0$ such that if
$|x - a| < \delta_1$, then $|f(x) - f(a)| < \frac{\epsilon}{2}$.

Since g is continuous at $a \in U$, we know there exists a $\delta_2 > 0$ such that if
$|x - a| < \delta_2$, then $|g(x) - g(a)| < \frac{\epsilon}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$:

If $|x - a| < \delta$, then $|f(x) - f(a)| < \frac{\epsilon}{2}$
and $|g(x) - g(a)| < \frac{\epsilon}{2}$

Thus,

$$\begin{aligned}\left|\left(f(x) + g(x)\right) - \left(f(a) + g(a)\right)\right| &= \left|\left(f(x) - f(a)\right) + \left(g(x) - g(a)\right)\right| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

and $f + g$ is continuous at $a \in U$.

Theorem: If $f: A \to \mathbb{R}^m$ is continuous, $A \subseteq \mathbb{R}^n$, and A is compact, then
$f(A) \subseteq \mathbb{R}^m$ is compact.