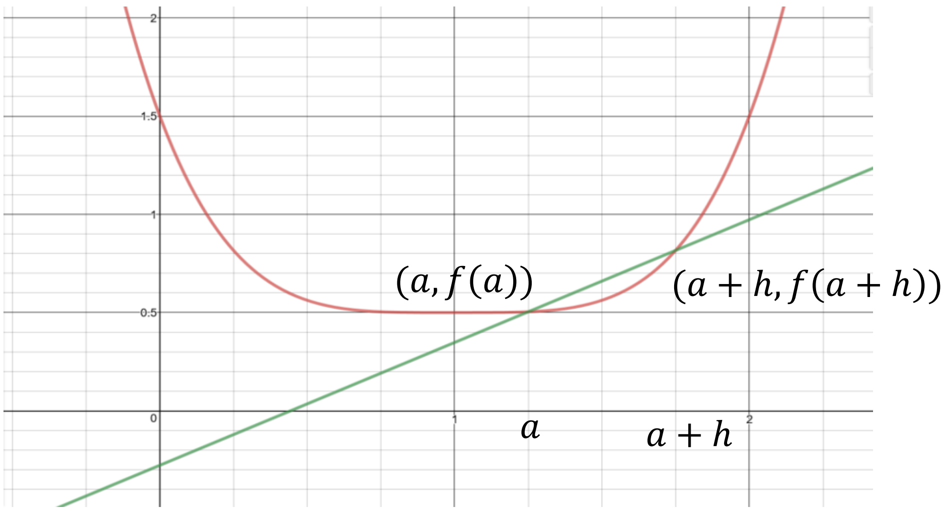
Partial Derivatives

Recall for a function of 1 variable that the definition of a derivative was:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

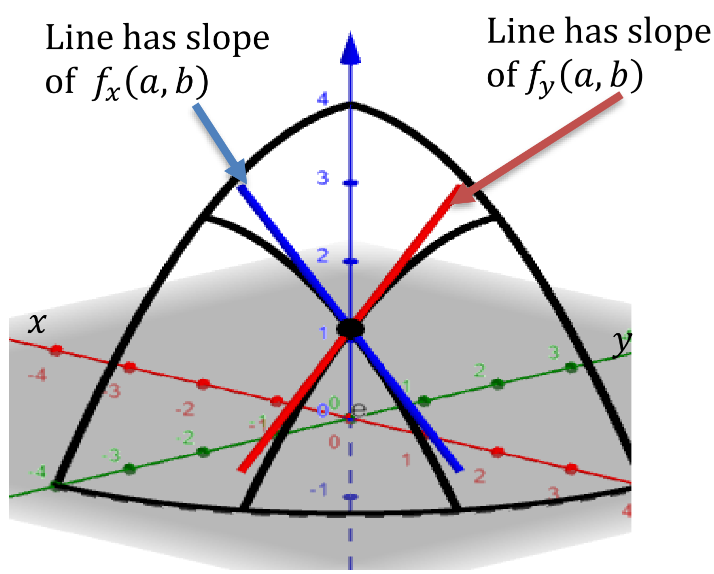
If the limit exists.



There are only 2 directions to approach a by, from the right or from the left. For a function of 2 variables there are an infinite number of directions we can approach a point (a, b).

However, there are 2 special sets of directions we can look at:

1. Let $y = b$ and let x approach a
2. Let $x = a$ and let y approach b.



$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

These are called partial derivatives of f with respect to x and y at (a, b).

Def. If f is a function of 2 variables, then the **partial derivatives, f_x and f_y**, are:

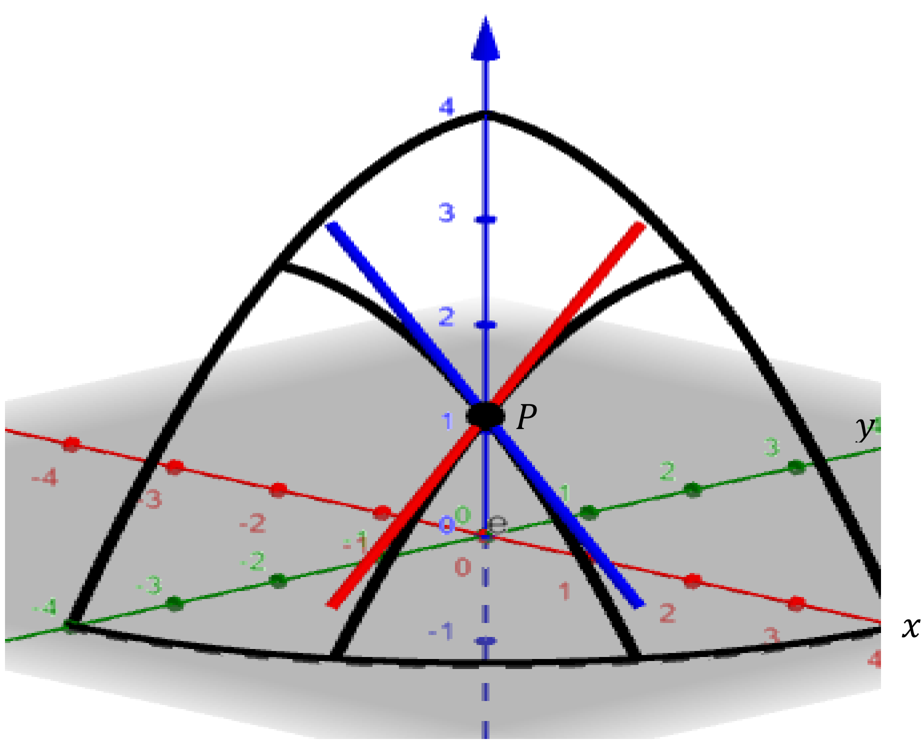
$$\boldsymbol{f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}}$$

$$\boldsymbol{f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}}$$

if the limits exist.

Just like $f'(x)$ gives you the rate of change of the value of a function $y = f(x)$, $f_x(x,y)$ gives the rate of change of the value of $f(x,y)$ in the x direction (holding y constant) and $f_y(x,y)$ gives the rate of change of the value of $f(x,y)$ in the y direction (holding x constant). So if $f_x(1,-2) > 0 \Rightarrow$ if you increase x a little from $x = 1, y = -2$, then the value of z increases.

Ex. In the example below, if you are at $P\big(a,b,f(a,b)\big)$, and you increase x and hold y constant, then the value of $f(x,b)$ decreases. If you increase y and hold x constant, then the value of $f(a,y)$ increases.



Notation: If $z = f(x, y)$, then we write:

$$f_x = D_1 f = f_1 = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x} = D_x f$$

$$f_y = D_2 f = f_2 = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y} = D_y f.$$

A partial derivative is an ordinary derivative of a single variable where we treat the second variable as a constant.

Ex. Let $f(x, y) = x^2 + 2x^3y^2 - x \sin y$. Find $f_x(2,0)$ and $f_y(2,0)$.

$$f_x(x, y) = 2x + 6x^2y^2 - \sin y$$

$$f_y(x, y) = 0 + 4x^3y - x \cos y$$

$$f_x(2,0) = 2(2) + 2(2)^2(0)^2 - \sin 0 = 4$$

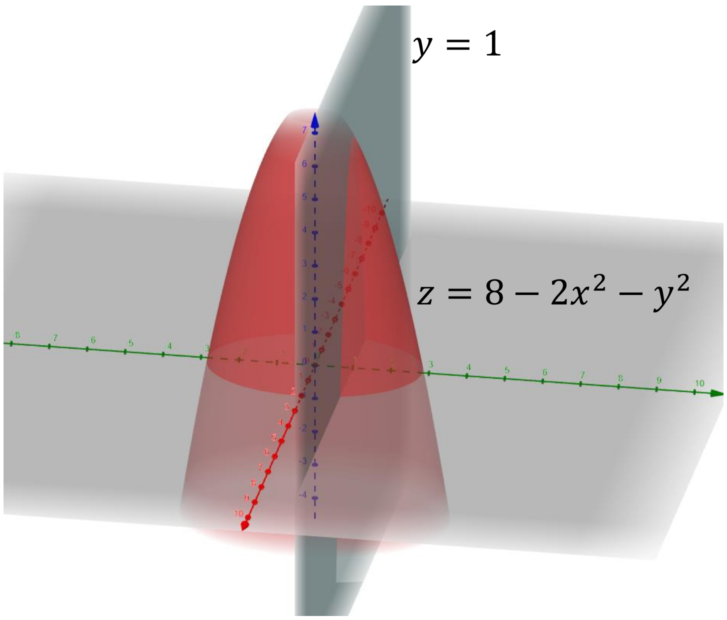
$$f_y(2,0) = 4(2)^3(0) - 2(\cos 0) = -2.$$

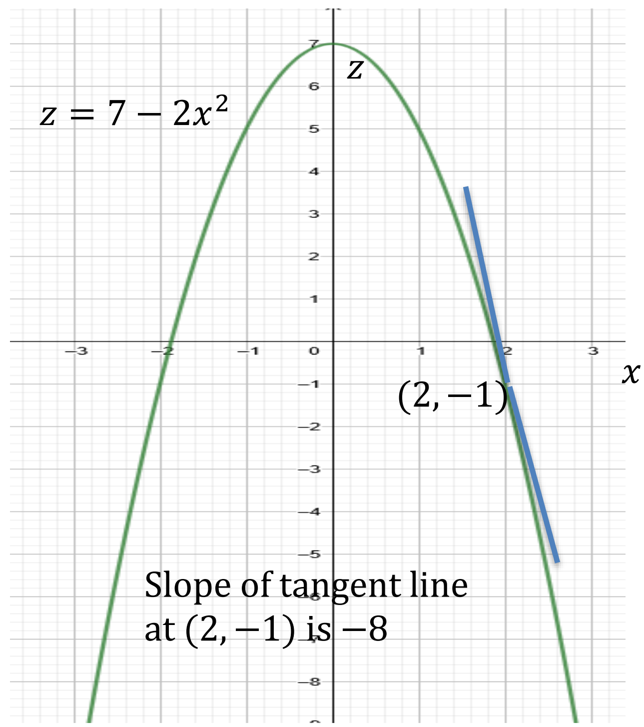
Ex. Let $f(x, y) = 8 - 2x^2 - y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and then interpret these numbers as slopes.

$$f_x(x, y) = -4x \qquad\qquad f_y(x, y) = -2y$$

$$f_x(2, 1) = -8 \qquad\qquad f_y(2,1) = -2$$

If we slice the paraboloid by the plane $y = 1$, then the intersection is the curve, $z = 8 - 2x^2 - (1)^2 = 7 - 2x^2$. For a parabola in the $(x, 1, z)$ plane $z = 7 - 2x^2$, the slope of the tangent line to that parabola at $(2,1,-1)$ is $f_x(2, 1) = -8$ (i.e. $f(x, y)$ is decreasing in the x direction at $(2, 1, -1)$).





If we slice the paraboloid by the plane $x = 2$, then we get a parabola:

$$z = 8 - (2)(2)^2 - y^2 = -y^2 \text{ at } (2, 1, -1) .$$

The slope of the tangent line to $z = -y^2$, at $(2,1,-1)$ is -2 (i.e. $f(x,y)$ is decreasing in the y direction at $(2, 1, -1)$.)

Ex. Chain rule: $f(x,y) = e^{xy} + (x^2 + y^2)^{10}$. Find f_x and f_y.

$$f_x = e^{xy}\frac{\partial}{\partial x}(xy) + 10(x^2 + y^2)^9\frac{\partial}{\partial x}(x^2 + y^2)$$

$$f_x = ye^{xy} + 10(x^2 + y^2)^9(2x)$$

$$f_x = ye^{xy} + 20x(x^2 + y^2)^9.$$

$$f_y = e^{xy}\frac{\partial}{\partial y}(xy) + 10(x^2 + y^2)^9\frac{\partial}{\partial y}(x^2 + y^2)$$

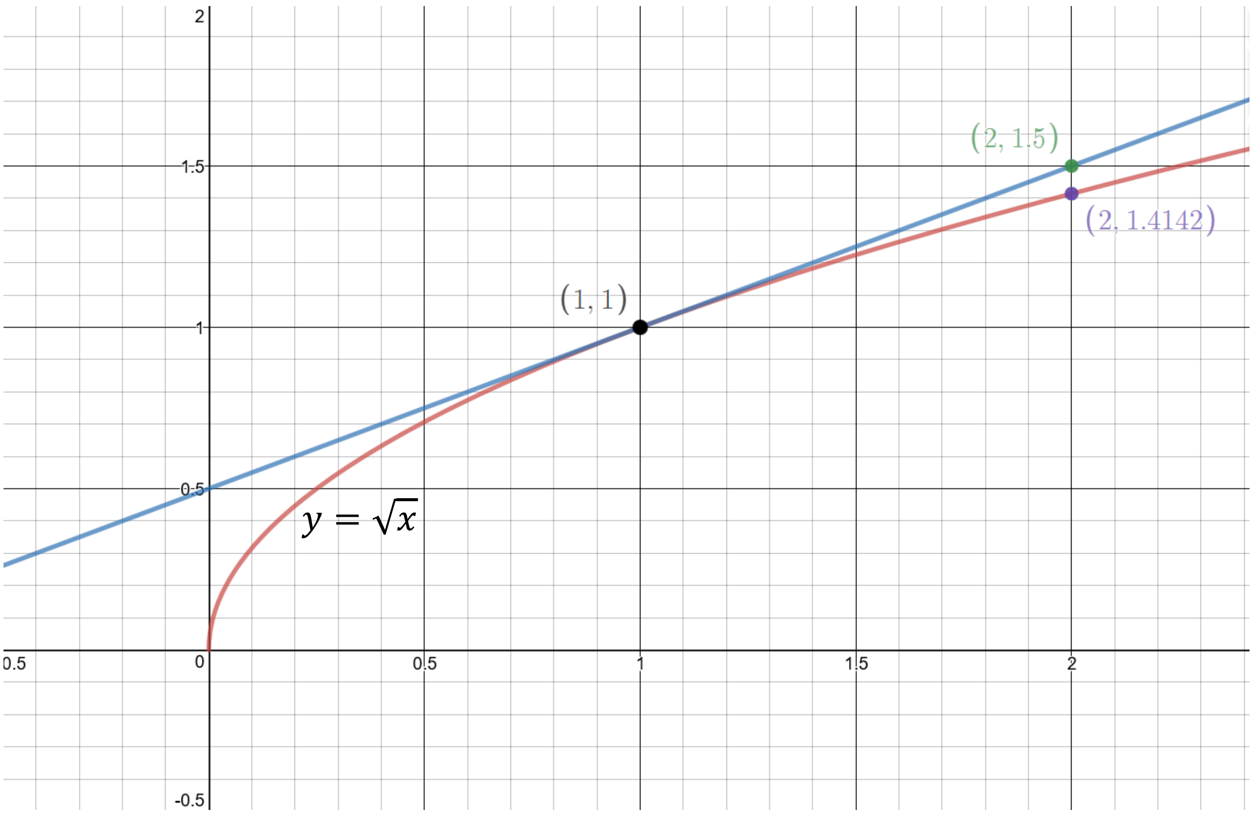
$$f_y = xe^{xy} + 10(x^2 + y^2)^9(2y)$$

$$f_y = xe^{xy} + 20y(x^2 + y^2)^9.$$

<u>Tangent Planes</u>

For functions of 1 variable, we found the equation of a tangent line to a curve. In particular, we could use the tangent line to approximate the value of a function.

Ex. Use the tangent line to the graph of $y = \sqrt{x}$ at the point $(1,1)$ to approximate $\sqrt{2}$.



To do this we need to find the equation of the tangent line at $(1,1)$ and then find the y value along the tangent line when $x = 2$.

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Slope of tangent line at $x = 1$ is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

Equation of tangent line at $x = 1$:

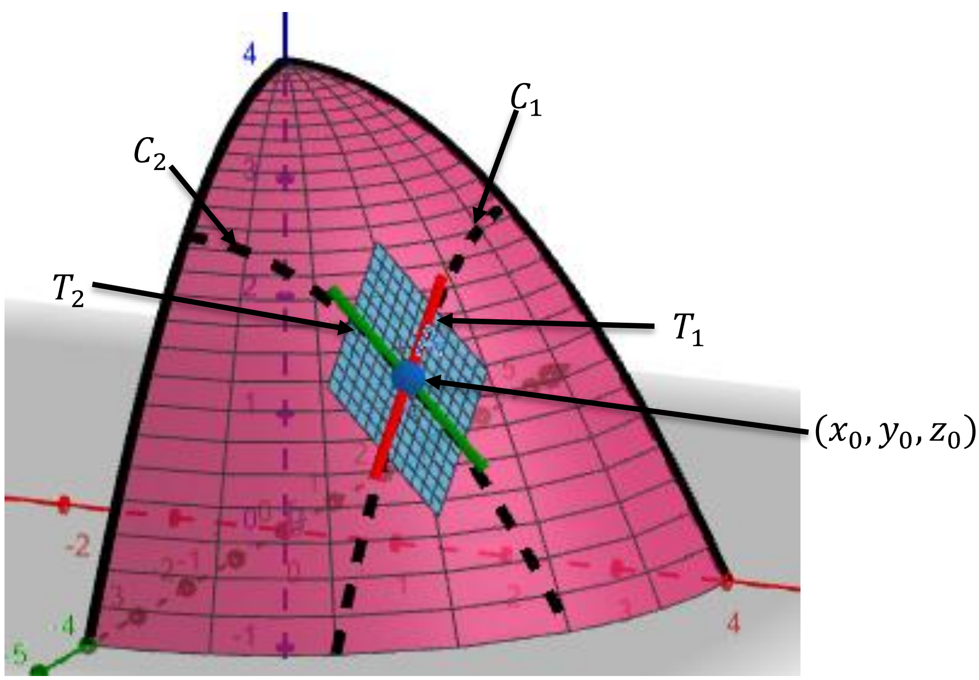
$$y - 1 = \frac{1}{2}(x - 1) \quad \text{or} \quad y = \frac{1}{2}(x - 1) + 1$$

$L(x) = \frac{1}{2}(x - 1) + 1$ is the **linear approximation of** $f(x) = x^{\frac{1}{2}}$ at $x = 1$.

So we can approximate $\sqrt{2}$ by:

$$\sqrt{2} \approx L(2) = \frac{1}{2}(2 - 1) + 1 = \frac{1}{2}(1) + 1 = 1.5.$$

For functions of 2 variables, the graphs are surfaces instead of curves and we have tangent planes instead of tangent lines. For $z = f(x, y)$, let (x_0, y_0, z_0) be on the surface. If we cut the surface with the plane $y = y_0$, then we can get a curve, C_1, and a tangent line, T_1 (in red). If we cut the surface with the plane $x = x_0$, then we get a curve, C_2, with a tangent line, T_2 (in green). The tangent plane is the plane that contains those 2 lines (in blue).



Actually, if C is any curve that lies on the surface through (x_0, y_0, z_0), then its tangent line will also be in that plane.

We know the equation of any plane through (x_0, y_0, z_0) is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$$

or

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

where $a = -\frac{A}{C}$, $b = -\frac{B}{C}$.

If we intersect this plane with the plane $y = y_0$, then we get:

$$z - z_0 = a(x - x_0); \qquad y = y_0$$

These two equations give us a line (the intersection of 2 planes) with a slope a. We know the slope of the tangent line, T_1, is $f_x(x_0, y_0)$. Therefore, if we start with the tangent plane with the equation:

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

then

$$a = f_x(x_0, y_0)\ .$$

Similarly, if we intersect the tangent plane with the plane $x = x_0$, we get the line:

$$z - z_0 = b(y - y_0); \quad x = x_0.$$

The slope of this line is b, which equals $f_y(x_0, y_0)$. Thus we have:

$$b = f_y(x_0, y_0).$$

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is:

$$\boldsymbol{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).}$$

Ex. Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2 + 1$ at the point $(1, -1, 4)$.

$$(x_0, y_0, z_0) = (1, -1, 4)$$

$$f_x = 2x \qquad\qquad f_y = 4y$$

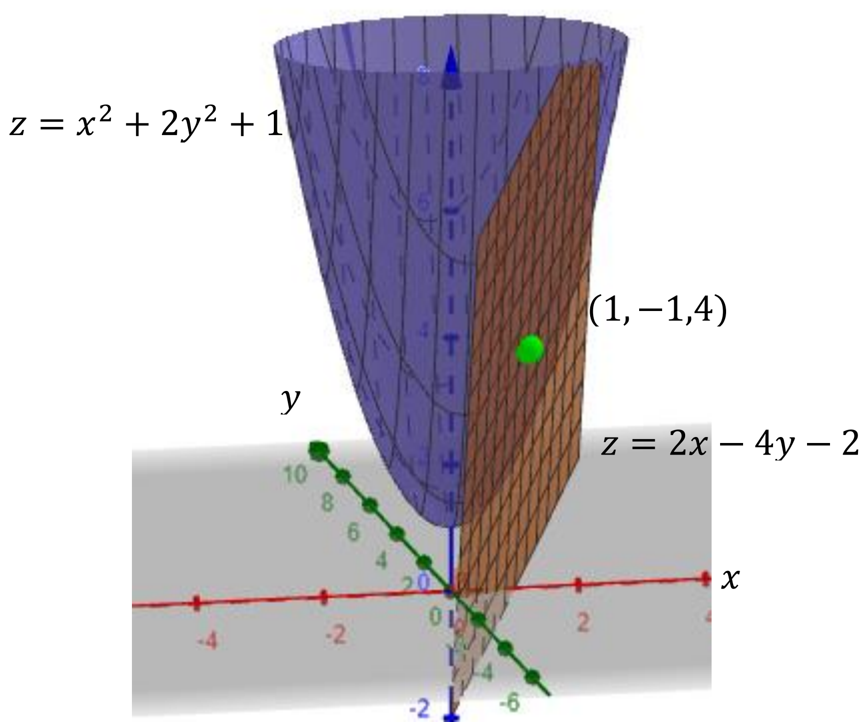
$$f_x(1, -1) = 2 \qquad\qquad f_y(1, -1) = -4$$

Equation of tangent plane at $(1, -1, 4)$:

$$z - 4 = 2(x - 1) - 4(y + 1)$$

$$z - 4 = 2x - 2 - 4y - 4$$

$$z = 2x - 4y - 2$$



Linear Approximations

Just as we used the tangent line to approximate the values of a curve near a point, we can use the tangent plane to approximate the values of a function of 2 variables. The equation of a plane is simple, but evaluating a complicated function can be hard.

In the last example we had the surface $z = x^2 + 2y^2 + 1$ (elliptic paraboloid) whose tangent plane at $(1, -1, 4)$ was $z = 2x - 4y - 2$.

$L(x, y) = 2x - 4y - 2$ is called the **linearization of *f*** at $(1, -1)$.
So $L(x, y) \approx f(x, y)$ when (x, y) is "not too far" from $(1, -1)$.

Ex. Approximate the value of $(1.05)^2 + 2(-1.1)^2 + 1$.

Let $z = f(x, y) = x^2 + 2y^2 + 1$.
We want to approximate $f(1.05, -1.1)$.
We can do this 2 different ways.

Approach 1: Using the formula:

$$z = f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

In this case, $x = 1.05, \quad y = -1.1, \quad a = 1, \quad b = -1.$

We know from the previous example that:

$f_x(1, -1) = 2 \qquad\qquad f_y(1, -1) = -4,$ so

$$f(1.05, -1.1) \approx f(1, -1) + 2(1.05 - 1) - 4(-1.1 - (-1)).$$

$$= 4 + 2(.05) - 4(-.1) = 4.5.$$

Approach 2: Find the z value of the point $(1.05, -1.1)$ on the tangent plane to $z = f(x, y)$ at $(1, -1)$.

In the previous example we found the equation of this tangent plane to be: $L(x, y) = 2x - 4y - 2$.

$$f(1.05, -1.1) \approx L(1.05, -1.1) = 2(1.05) - 4(-1.1) - 2$$
$$= 4.5.$$

Notice that the equation of the tangent plane is:

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Using the tangent plane in this form is exactly approach #1.

Notice we can define partial derivatives for functions of 3 (or more) variables:

$$w = f(x, y, z)$$

$$\frac{\partial f}{\partial z} = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h}$$

Ex. Let $f(x, y, z) = e^{xy} \sin(y^2 z)$. Find f_x, f_y, and f_z.

$$f_x = ye^{xy} \sin(y^2 z)$$

$$f_y = e^{xy}((\cos(y^2 z))2yz) + (\sin(y^2 z))(xe^{xy})$$

$$f_z = e^{xy}(\cos(y^2 z))\, y^2.$$

A linear approximation can be defined for more than 2 variables.
If we have $w = f(x, y, z)$, then we write:

$$f(x, y, z) \approx L(x, y, z)$$
$$= f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

Ex. Find the linear approximation, $L(x, y, z)$, of the function $V = xyz$ at the point $(1, 2, 4)$, and approximate the value of $(0.95)(2.01)(4.1) = V(0.95, 2.01, 4.1)$.

$$V_x = yz \qquad V_x = (1, 2, 4) = 8$$
$$V_y = xz \qquad V_y = (1, 2, 4) = 4$$
$$V_z = xy \qquad V_z = (1, 2, 4) = 2$$

$$V(x, y, z) \approx L(x, y, z)$$

$$= V(1, 2, 4) + V_x(1, 2, 4)(x - 1) + V_y(1, 2, 4)(y - 2) + V_z(1, 2, 4)(z - 4).$$

$$= 8 + 8(x - 1) + 4(y - 2) + 2(z - 4).$$

$$V(0.95, 2.01, 4.1) \approx 8 + 8(0.95 - 1) + 4(2.01 - 2) + 2(4.1 - 4)$$
$$= 7.84.$$

Or we could have combined terms in $L(x, y, z)$ and then plugged in.

$$L(x, y, z) = 8 + 8(x - 1) + 4(y - 2) + 2(z - 4)$$
$$= 8x + 4y + 2z - 16.$$

$$V(0.95, 2.01, 4.1) \approx L(0.95, 2.01, 4.1)$$
$$= 8(0.95) + 4(2.01) + 2(4.1) = 7.84.$$

<u>Differentiability: The General Case</u>

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ then we define the derivative, $Df(\overrightarrow{x_0})$, to be:

$$Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $f(\overrightarrow{x_0}) = < f_1(\overrightarrow{x_0}), \dots, f_m(\overrightarrow{x_0}) >$.

Ex. Let $f(x,y) = (e^{x+y} + y,\ y^2 x)$, find the following:

a. $Df(x,y)$
b. $Df(0,1)$

a. $Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x+y} & e^{x+y} + 1 \\ y^2 & 2xy \end{bmatrix}$

b. $Df(0,1) = \begin{bmatrix} e & e+1 \\ 1 & 0 \end{bmatrix}$.

Def: Let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$, then **the gradient of $\boldsymbol{f}$**, $\boldsymbol{\nabla f}$, is

$$\boldsymbol{\nabla f} =< \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial x_1}}, \dots, \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial x_n}} >.$$

In particular, If $f: \mathbb{R}^3 \to \mathbb{R}$, then:

$$\nabla f =< \frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial f}{\partial z} > = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$$

Ex. Suppose $f(x, y, z) = xe^y + z$, find $\nabla f(1, 0, 1)$.

$$\nabla f(x, y, z) =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > = < e^y, xe^y, 1 > = (e^y)\vec{\imath} + (xe^y)\vec{\jmath} + \vec{k}$$

$$\nabla f(1, 0, 1) = < e^0, 1e^0, 1 > = < 1, 1, 1 > = \vec{\imath} + \vec{\jmath} + \vec{k}.$$

Ex. Suppose $f(x, y) = e^{xy} + \cos(xy)$; find $\nabla f(x, y)$ and $\nabla f(0,1)$.

$$\nabla f(x, y) =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > = < ye^{xy} - y\sin(xy), \ \ xe^{xy} - x\sin(xy) >$$

$$= (ye^{xy} - y\sin(xy))\vec{\imath} + (xe^{xy} - x\sin(xy))\vec{\jmath}.$$

$$\nabla f(0,1) = \big(1(1) - 1(0)\big)\vec{\imath} + \big(0(e^0) - 0(0)\big)\vec{\jmath} = \vec{\imath}.$$