Graphing Functions of 2 Variables in $\mathbb{R}^3$

Many things depend on more than 1 unknown (e.g. the cost of building a car, volume of circular cylinder: $V = \pi r^2 h$, etc).

Def. A **real-valued function of 2 variables** (or n variables) is a rule that assigns to each point (x, y) in a domain, D, a real number denoted by $f(x, y)$. The set of values of f is called the range $= \{f(x, y) | (x, y) \in D\}$.

We often write: $z = f(x, y)$

Ex. $f(x, y) = x^2 + y^2 - 3e^x$ is a real valued function of 2 variables.

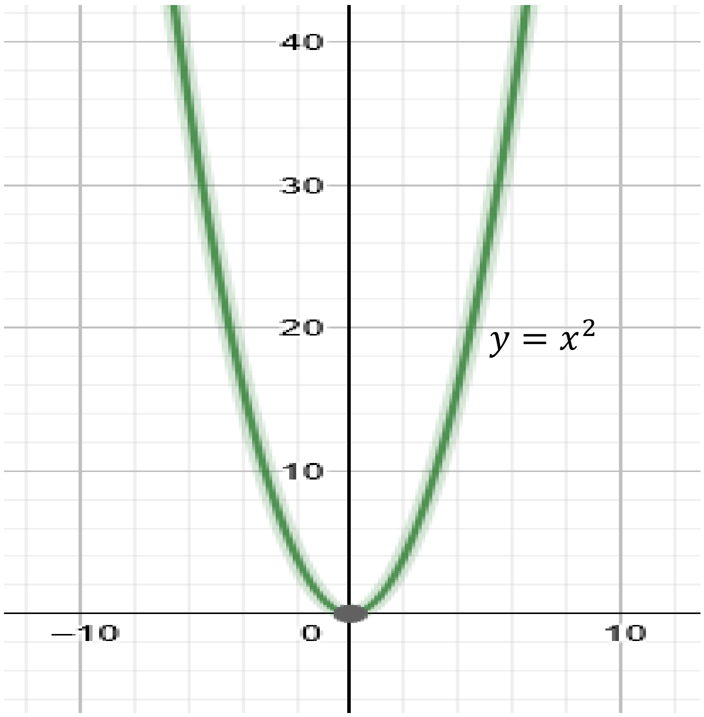
Def. A **vector-valued function of 2 variables** (or n variables) is a rule that assigns each point (x, y) to a vector in $\mathbb{R}^m$, $m > 1$.

Ex. $f(x, y) =< x^2 + y^2, \cos x, \sin y >$ is a vector valued function of 2 variables.

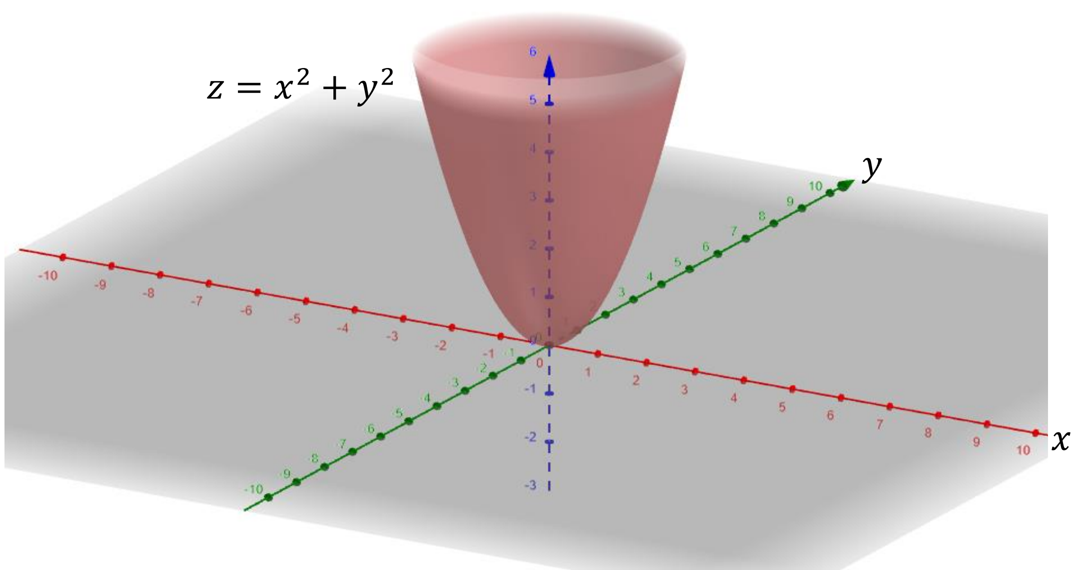
Ex. $f(t) =< \cos t, \sin t >$, is a vector valued function of 1 variable.

Def. If f is a real valued function of 2 variables with domain, D, then the **graph of f** is the points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$, $(x, y) \in D$.

The graph of $y = f(x)$ is a curve in $\mathbb{R}^2$



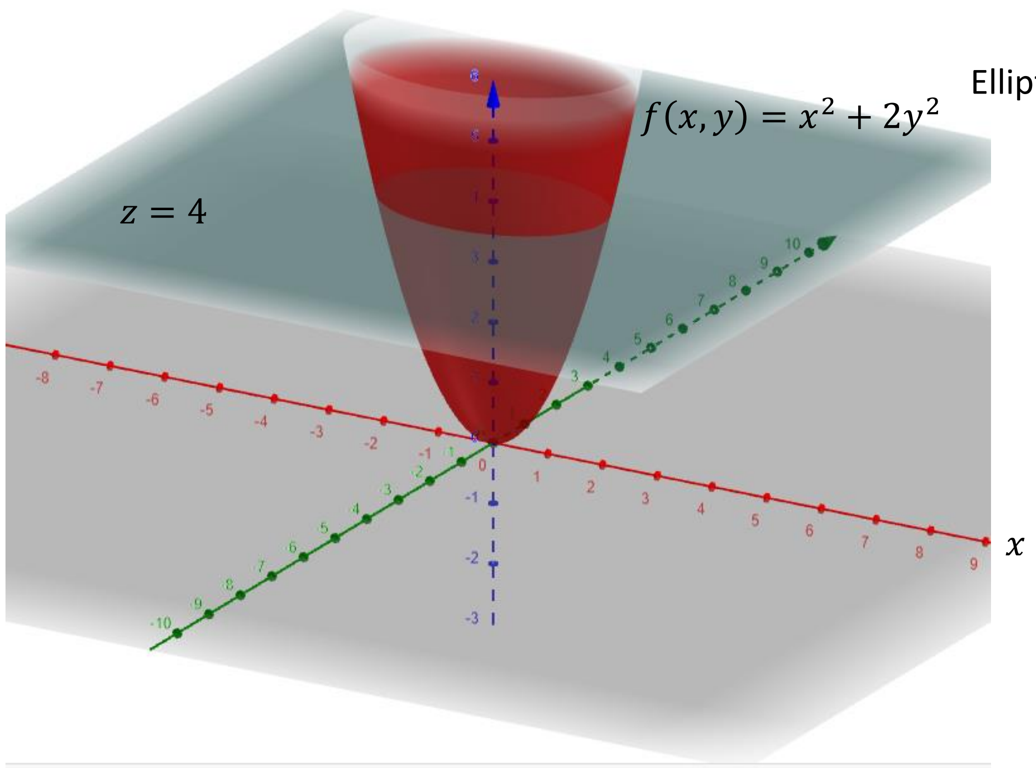
The graph of $z = f(x, y)$ is a surface in $\mathbb{R}^3$



Def. Level curves and surfaces: let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ and let $k \in \mathbb{R}$, then the **level set of value k** is defined to be the set of points $\vec{x} \in U$, at which $f(\vec{x}) = k$. If $n = 2$, then we call the level set a **level curve** and if $n = 3$ we call it a **level surface**.

Ex. Sketch $f(x, y) = x^2 + 2y^2$. Domain $= \mathbb{R}^2$; Range $= z \geq 0$.

Level curves for $z = k > 0$ are ellipses.



Def. A **section of the graph of f** is the intersection of the graph of f with a vertical plane (i.e., a plane parallel to the xz or yz plane).

Ex. Let $z = f(x, y) = x^2 + 2y^2$

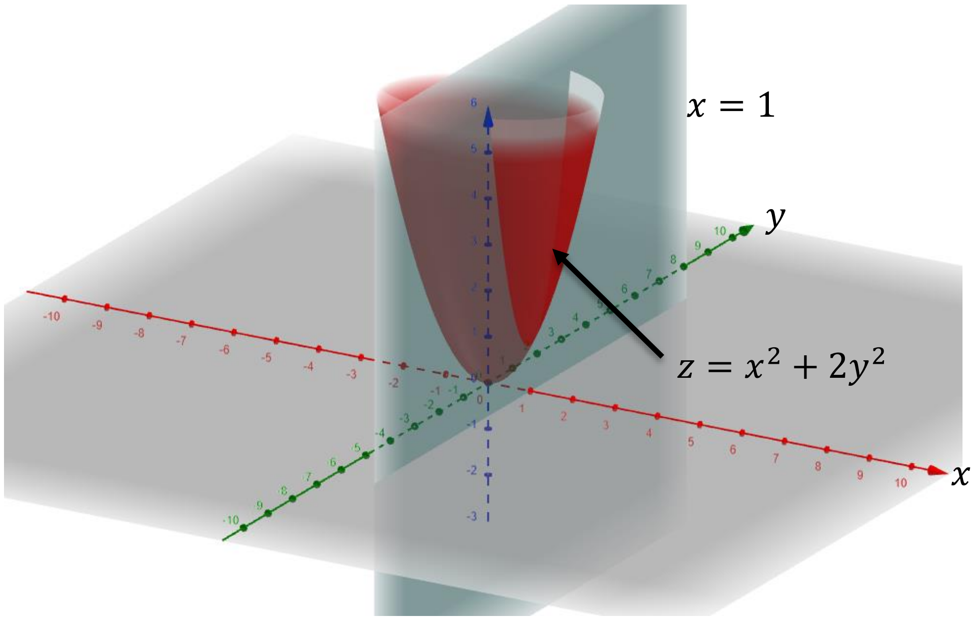
Sections of the graph of f are parabolas.

For example, if $y = k$, then we get:

$$z = x^2 + 2k^2$$

If $x = k$, then we get:

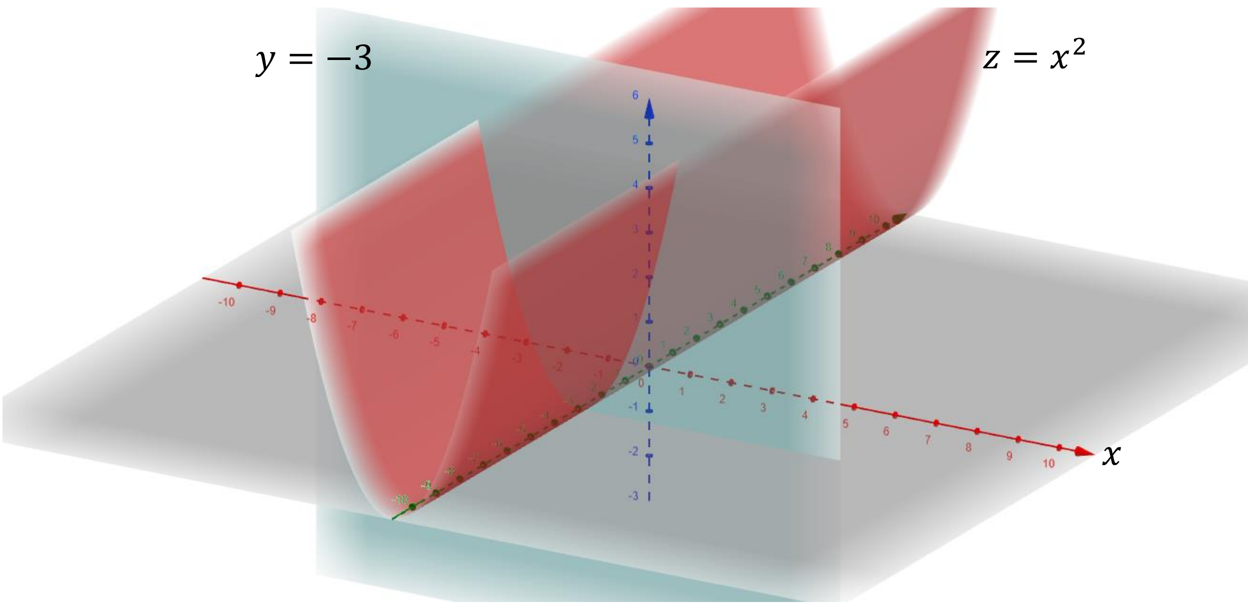
$$z = k^2 + 2y^2$$



Cylinders: a **cylinder** consists of all lines (called rulings) that are parallel to a given line and pass through a given plane.

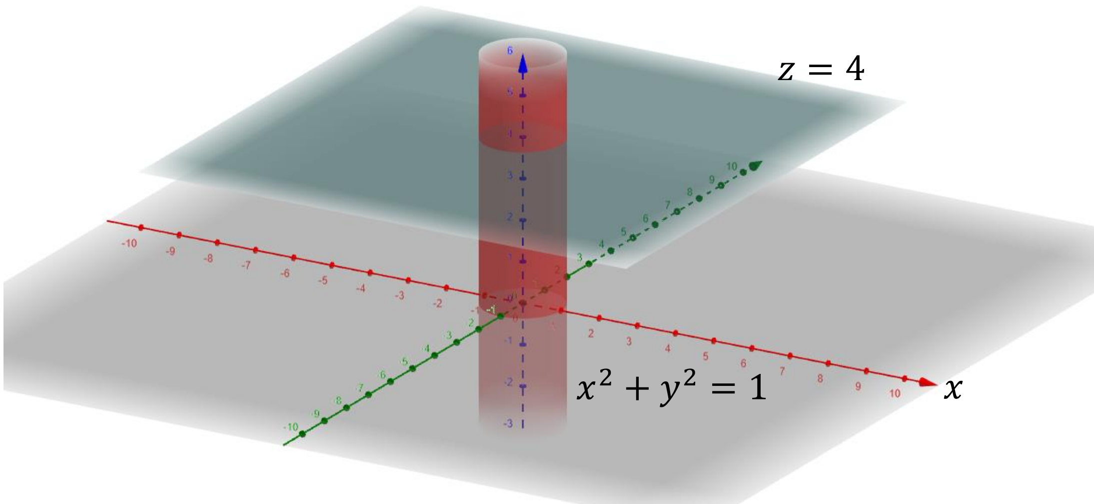
Ex. Sketch the surface $z = x^2$.

In the xz plane ($y = 0$) this is just the parabola $z = x^2$. Since the function does not have a "y" in it, every cross sectional of the plane $y = k$ is the same parabola. This is called a **parabolic cylinder**. In fact, if one of x, y, z is missing from the equation, then you will get a cylinder.

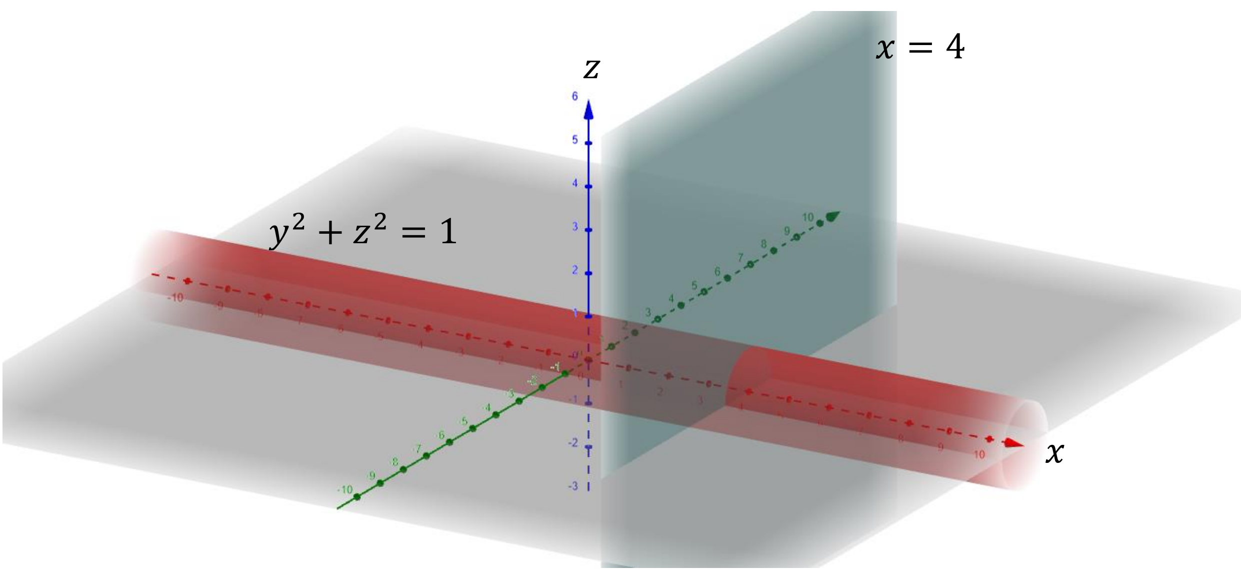


Ex. Sketch in $\mathbb{R}^3$: a) $x^2 + y^2 = 1$ b) $y^2 + z^2 = 1$

a) $x^2 + y^2 = 1$ is a circle of radius 1 in $z = k$ plane.



b) $y^2 + z^2 = 1$ is a circle of radius 1 in $x = k$ plane.



Ex. Use the level curves and sections to sketch $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$

When $z = 0$: $x^2 + \frac{y^2}{9} = 1$

$z = k$: $x^2 + \frac{y^2}{9} + \frac{k^2}{4} = 1$

$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$ is an ellipse if $-2 < k < 2$. As $k \to 2$ or -2 the major and minor axes are shrinking to 0. For example,

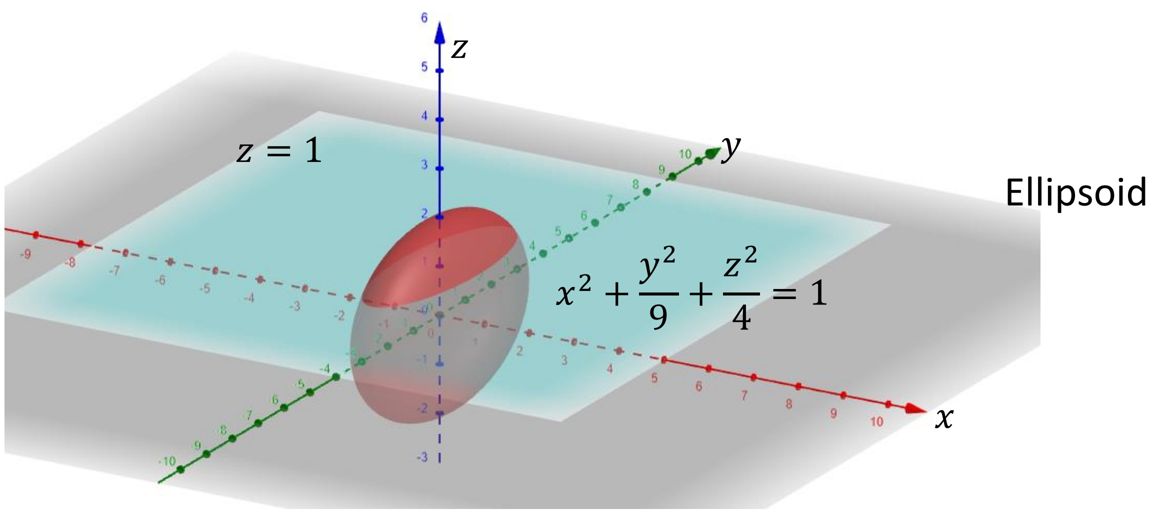
$k = 1$: $x^2 + \frac{y^2}{9} = \frac{3}{4} \Rightarrow \frac{x^2}{\frac{3}{4}} + \frac{y^2}{\frac{27}{4}} = 1.$

At $x = 0$: $\frac{y^2}{9} + \frac{z^2}{4} = 1$ ellipse in yz plane

At $x = k$: $\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2$ $\quad -1 < k < 1$ ellipse

At $y = 0$: $x^2 + \frac{z^2}{4} = 1$ ellipse in the xz plane

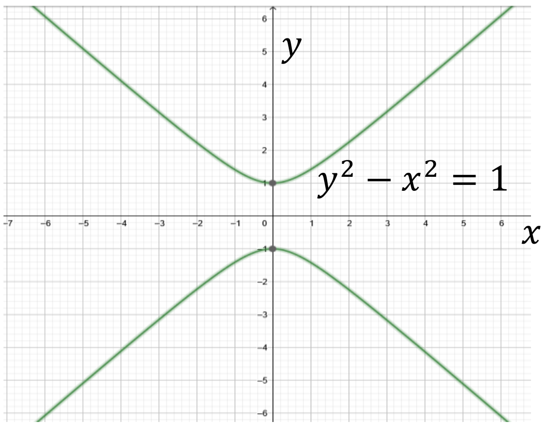
At $y = k$: $x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9}$ $\quad -3 < k < 3$ ellipse



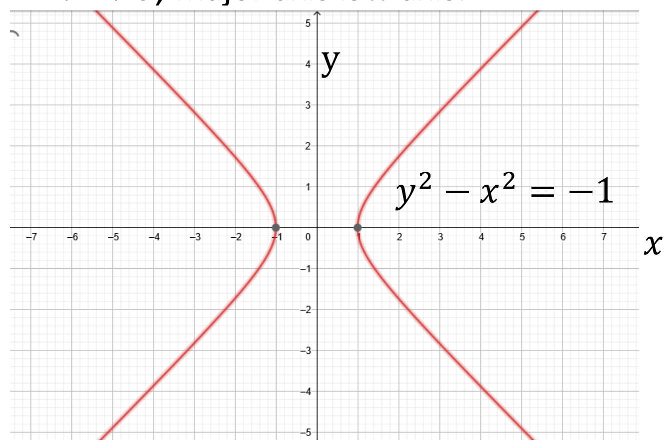
Ex. Use level curves and sections to sketch $z = y^2 - x^2$.

Level curves: $k = y^2 - x^2$:

Hyperbolas, $k > 0$, major axis is y axis

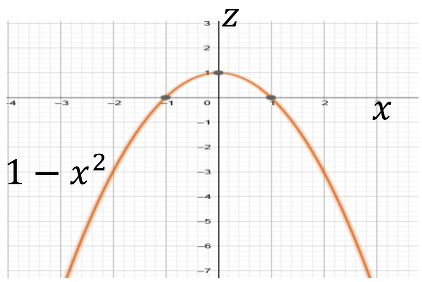


$k < 0$, major axis is x axis.



Sections:

$z = 1 - x^2$

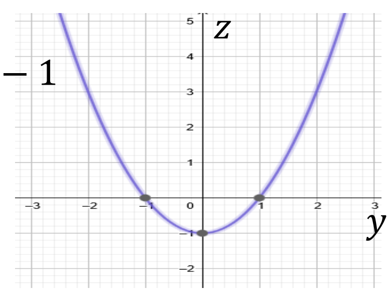


$y = k; \quad z = k^2 - x^2,$

parabolas in xz plane opening

in negative z direction.

$z = y^2 - 1$

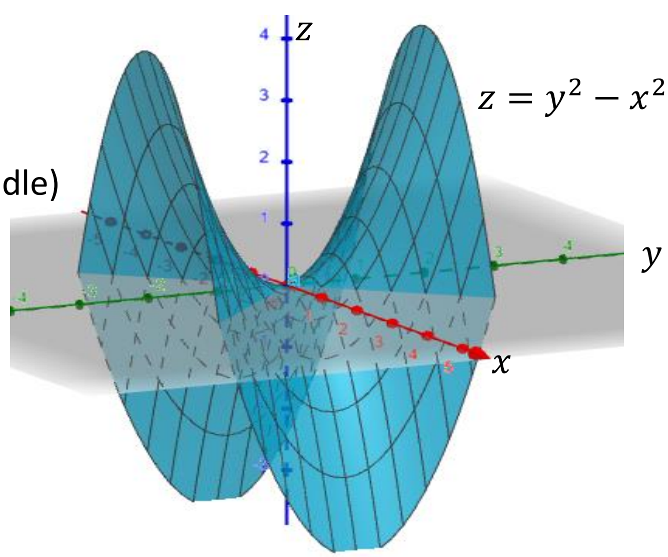


$x = k; \quad z = y^2 - k^2,$

parabolas in yz plane opening

opening in positive z direction.

Hyperbolic Paraboloid (Saddle)



Ex. Sketch $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$, using level curves and sections.

At $z = k$: $\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4}$ ellipse in slices II xy plane

At $y = 0$: $\frac{x^2}{4} - \frac{z^2}{4} = 1$ hyperbola in slices II xz plane

At $y = k$: $\frac{x^2}{4} - \frac{z^2}{4} = 1 - k^2$, hyperbolas in xz plane if $k \neq \pm 1$

$-1 < k < 1$, major axis is x axis

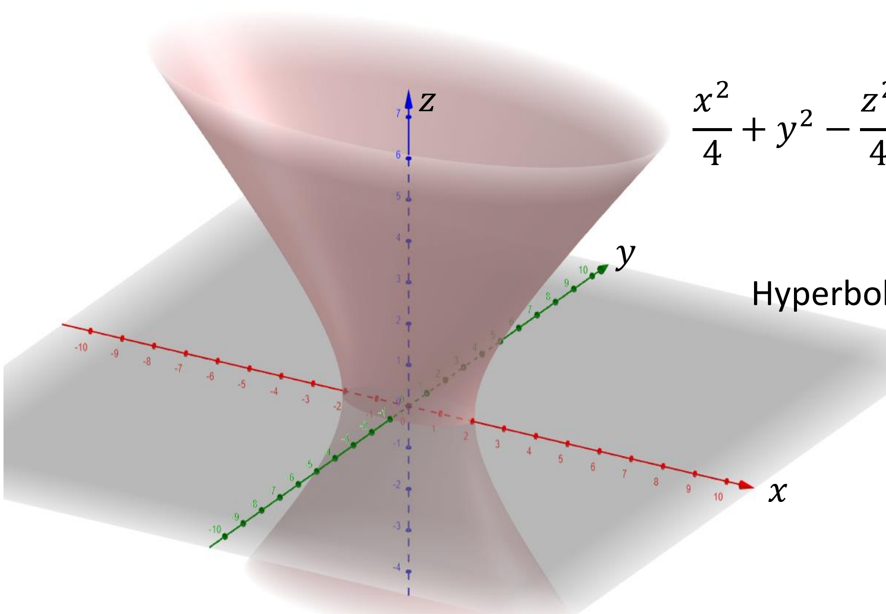
$k < -1$ or $k > 1$, major axis is z axis.

At $x = 0$: $y^2 - \frac{z^2}{4} = 1$ hyperbolas in slices II to yz plane

At $x = k$: $y^2 - \frac{z^2}{4} = 1 - \frac{k^2}{4}$, hyperbolas in yz plane if $k \neq \pm 2$

$-2 < k < 2$, major axis is y axis

$k < -2$ or $k > 2$, major axis is z axis.



$$\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$$

Hyperboloid of one sheet

If $-\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$, then the major axis is the x-axis (with negative term).

Ex. sketch $z^2 = \frac{x^2}{2} + \frac{y^2}{3}$, using level curves and sections.

$z = k$: $k^2 = \frac{x^2}{2} + \frac{y^2}{3}$ slices II to xy plane are ellipses if $k \neq 0$

if $k = 0$, then it's a point

$x = k$: $z^2 - \frac{y^2}{3} = \frac{k^2}{2}$ slices II to yz plane are hyperbolas if $k \neq 0$, major axis is the z axis.

$y = k$: $z^2 - \frac{x^2}{2} = \frac{k^2}{3}$ slices II to xz plane are hyperbolas if $k \neq 0$, major axis is the z axis.