Applications of Multiple Integrals

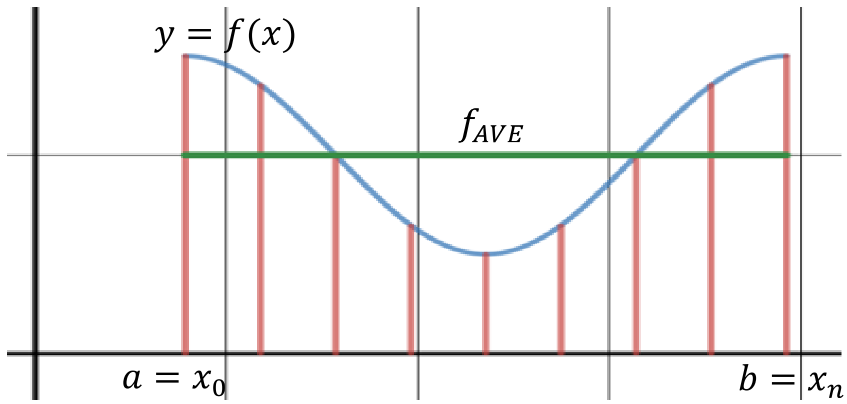
Recall from first year calculus that the average value of a real-valued function, $f(x)$, on an interval $[a, b]$ is defined to be:

$$f_{AVE} = \frac{1}{b-a}\int_a^b f(x)dx$$

This comes from taking the value of the function $f(x)$ at n equally spaced points on $[a, b]$, averaging them and taking a limit as n goes to ∞.

$$\frac{f(x_1)+\cdots+f(x_n)}{n}$$

$$\Delta x = \frac{b-a}{n}$$



$$\frac{f(x_1)+\cdots+f(x_n)}{n} = \big(f(x_1) + \cdots + f(x_n)\big)\frac{\Delta x}{b-a}$$

$$f_{AVE} = \lim_{n\to\infty}\left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{1}{b-a}\int_a^b f(x)\;dx$$

Notice that $b - a = \int_a^b dx$, so we could write:

$$f_{AVE} = \frac{\int_a^b f(x)\,dx}{\int_a^b dx}.$$

Similarly, we define the average of a real-valued function over a region $D \subseteq \mathbb{R}^2$ or a region $W \subseteq \mathbb{R}^3$ by:

$$f_{AVE} = \frac{\iint_D f(x,y)dx\,dy}{\iint_D dx\,dy}$$

$$f_{AVE} = \frac{\iiint_W f(x,y,z)dx\,dy\,dz}{\iiint_W dx\,dy\,dz}$$

Notice that the denominators are the area of D and the volume of W, respectively.

Ex. Find the average value of $f(x,y) = x\sin(xy)$ on the region,

$$D = \left[0, \frac{\pi}{2}\right] \times [0, \pi].$$

$$\iint_D f(x,y)dx\,dy = \int_{y=0}^{y=\pi} \int_{x=0}^{x=\frac{\pi}{2}} (x\sin(xy))\ dx\,dy$$

This is much easier to calculate if we reverse the order of integration:

$$= \int_{x=0}^{x=\frac{\pi}{2}} \int_{y=0}^{y=\pi} (x\sin(xy))\ dy\,dx = \int_{x=0}^{x=\frac{\pi}{2}} -\cos(xy)\Big|_{y=0}^{y=\pi} dx$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} (-\cos(\pi x) + 1)\ dx = \frac{-\sin(\pi x)}{\pi} + x\Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \frac{-\sin\left(\frac{\pi^2}{2}\right)}{\pi} + \frac{\pi}{2}$$

$$f_{AVE} = \frac{\iint_D \ f(x,y)dx\,dy}{\iint_D \ dx\,dy}$$

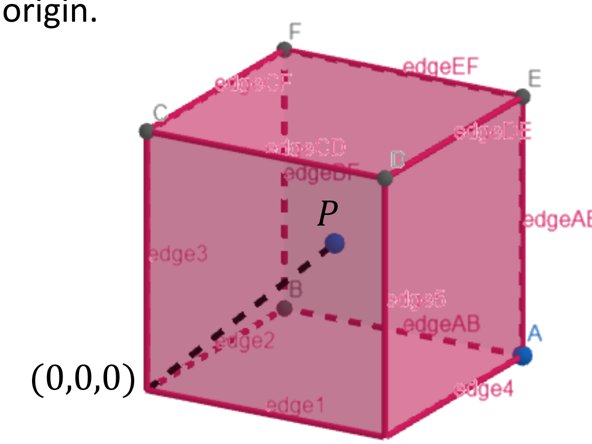
Since D is just a rectangle, its area (the denominator) is just:

$$\left(\frac{\pi}{2}\right)(\pi) = \frac{\pi^2}{2}$$

So we can write:

$$f_{AVE} = \frac{\frac{-\sin\left(\frac{\pi^2}{2}\right)}{\pi} + \frac{\pi}{2}}{\frac{\pi^2}{2}} = \frac{-2\sin\left(\frac{\pi^2}{2}\right)}{\pi^3} + \frac{1}{\pi}.$$

Ex. The temperature at points in the cube $W = [0,1] \times [0,1] \times [0,1]$ is proportional to the square of the distance to the origin. Find the average temperature.



$$T(x,y,z) = c(x^2 + y^2 + z^2)$$

$$T_{AVE} = \frac{\iiint_W c(x^2 + y^2 + z^2)dx\,dy\,dz}{\iiint_W dx\,dy\,dz}$$

The denominator is just the volume of W, i.e. 1.

$$T_{AVE} = c\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz$$

$$= c\int_0^1 \int_0^1 \frac{x^3}{3} + xy^2 + xz^2\Big|_0^1 dy\,dz = c\int_0^1 \int_0^1 \left(\frac{1}{3} + y^2 + z^2\right)dy\,dz$$

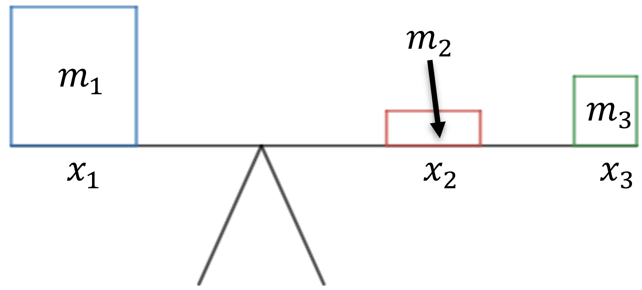
$$= c\int_0^1 \frac{1}{3}y + \frac{1}{3}y^3 + yz^2\Big|_0^1 dz = c\int_0^1 \left(\frac{1}{3} + \frac{1}{3} + z^2\right)dz$$

$$= c\left(\frac{2}{3}z + \frac{1}{3}z^3\right)\Big|_0^1 = c\left(\frac{2}{3} + \frac{1}{3}\right) = c\,.$$

Center of Mass

If masses $m_1, \dots, m_n$ are placed at $x_1, \dots, x_n$ on the x-axis, then the center of mass is defined to be:

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$



For a continuous mass density $\delta(x)$ along a lever the analogous formula is:

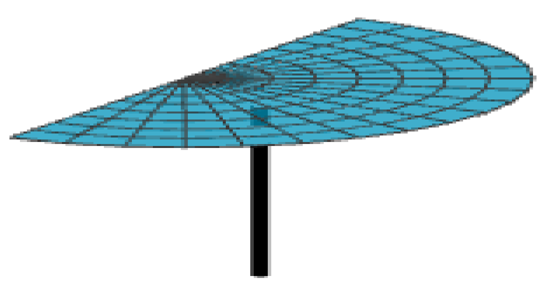
$$\bar{x} = \frac{\int_a^b x\, \delta(x)\, dx}{\int_a^b \delta(x) dx}$$

where $\int_a^b \delta(x) dx$ is the total mass.

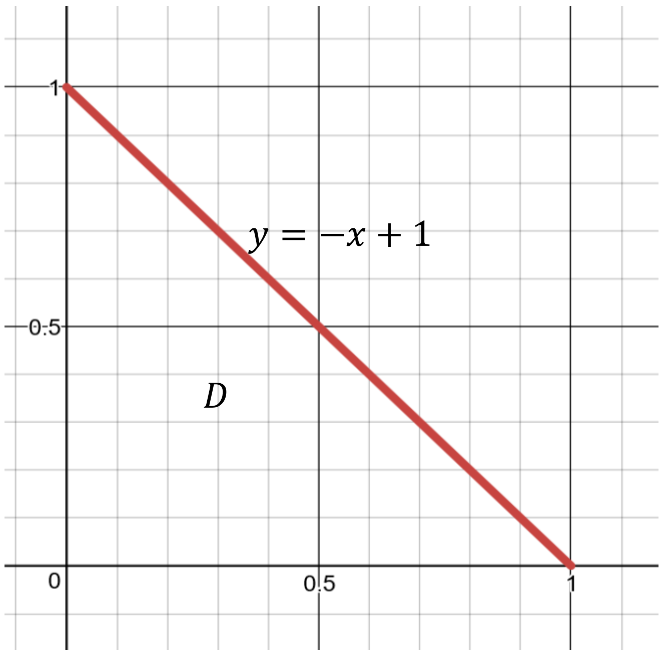
For two-dimensional plates, this generalizes to:

$$\bar{x} = \frac{\iint_D \ x\, \delta(x,y)\, dx\, dy}{\iint_D \ \delta(x,y)\, dx\, dy} \qquad \bar{y} = \frac{\iint_D \ y\, \delta(x,y)\, dx\, dy}{\iint_D \ \delta(x,y)\, dx\, dy}$$

where $\int_a^b \delta(x,y) dx\, dy$ is the mass of the plate.



Ex. Find the center of mass of a plate in the shape of a triangle with vertices at $(0,0), (1,0)$ and $(0,1)$ and a density function of $\delta(x,y) = xy$.



$$\text{mass} = m = \iint_D \delta(x,y)\,dx\,dy = \int_{x=0}^{x=1} \int_{y=0}^{y=-x+1} xy\,dy\,dx$$

$$= \int_{x=0}^{x=1} \frac{xy^2}{2}\Bigg|_{y=0}^{y=-x+1} dx = \int_{x=0}^{x=1} \frac{x}{2}(-x+1)^2\,dx$$

$$= \frac{1}{2}\int_{x=0}^{x=1} x(x^2 - 2x + 1)\,dx = \frac{1}{2}\int_{x=0}^{x=1} x^3 - 2x^2 + x\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{1}{24}.$$

$$\bar{x} = \frac{1}{m}\iint_D x\,\delta(x,y)\,dy\,dx = 24\left[\int_{x=0}^{x=1}\int_{y=0}^{y=-x+1} x^2 y\,dy\,dx\right]$$

$$= 24\left[\int_{x=0}^{x=1}\frac{x^2y^2}{2}\right]_{y=0}^{y=-x+1} dx = 12\int_{x=0}^{x=1} x^2(-x+1)^2\ dx$$

$$= 12\int_{x=0}^{x=1} x^2(x^2-2x+1)\ dx = 12\int_{x=0}^{x=1} x^4-2x^3+x^2\ dx$$

$$= 12\left(\frac{x^5}{5}-\frac{x^4}{2}+\frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{5}.$$

$$\bar{y} = \frac{1}{m}\iint_D y\,\delta(x,y)\,dy\,dx = 24\left[\int_{x=0}^{x=1}\int_{y=0}^{y=-x+1} xy^2\,dy\,dx\right]$$

$$= 24\left[\int_{x=0}^{x=1}\frac{xy^3}{3}\right]_{y=0}^{y=-x+1} dx = 8\int_{x=0}^{x=1} x(-x+1)^3\ dx$$

Either multiply out or let $u = -x+1$ and $du = -dx$:
when $x = 0, u = 1$; $x = 1, u = 0$

$$= -8\int_{u=1}^{u=0}(1-u)u^3\,du = -8\int_{u=1}^{u=0} u^3-u^4\,du$$

$$= -8\left(\frac{u^4}{4}-\frac{u^5}{5}\right)\Big|_1^0 = 8\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{2}{5}.$$

Center of mass: $(\bar{x},\bar{y}) = \left(\frac{2}{5},\frac{2}{5}\right)$

Notice that the problem is symmetric in x and y, so $\bar{x} = \bar{y}$.

For a region $W \subseteq \mathbb{R}^3$ with mass density $\delta(x, y, z)$:

$$mass = \iiint_W \delta(x, y, z)\, dx\, dy\, dz$$

And the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is:

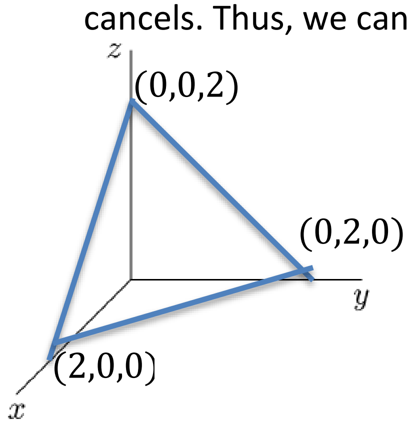
$$\bar{x} = \frac{\iiint_W x\, \delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}$$

$$\bar{y} = \frac{\iiint_W y\, \delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}$$

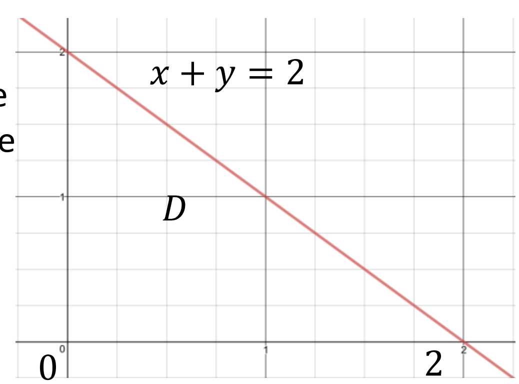
$$\bar{z} = \frac{\iiint_W z\, \delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}.$$

Ex. Find the center of mass of the region bounded by $x + y + z = 2$, $x = 0, y = 0$, and $z = 0$ assuming the density is uniform.

The density is uniform means the density is constant. Since the center of mass formulas have this constant in the numerator and denominator, it cancels. Thus, we can take $\delta(x, y, z) = 1$.



Projection into the x-y plane



$$\text{mass} = m = \int_{x=0}^{x=2} \int_{y=0}^{y=2-x} \int_{z=0}^{z=2-x-y} dz\, dy\, dx$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=2-x} z\Big|_{z=0}^{z=2-x-y} dy\, dx = \int_{x=0}^{x=2} \int_{y=0}^{y=2-x} (2-x-y)\ dy\, dx$$

$$= \int_{x=0}^{x=2} 2y - xy - \frac{y^2}{2}\Big|_{y=0}^{y=2-x} dx = \int_{x=0}^{x=2} 2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\ dx$$

$$= \int_{x=0}^{x=2} [(2-x)(2-x) - \frac{(2-x)^2}{2}]\, dx = \frac{1}{2}\int_{x=0}^{x=2} (2-x)^2\, dx$$

$$= \frac{1}{2}\int_{x=0}^{x=2} (4-4x+x^2)\ dx = \frac{4}{3}.$$

$$\bar{x} = \frac{1}{m}\int_{x=0}^{x=2} \int_{y=0}^{y=2-x} \int_{z=0}^{z=2-x-y} x\, dz\, dy\, dx$$

$$= \frac{3}{4}\int_{x=0}^{x=2} \int_{y=0}^{y=2-x} xz\Big|_{z=0}^{z=2-x-y} dy\, dx = \frac{3}{4}\int_{x=0}^{x=2} \int_{y=0}^{y=2-x} x(2-x-y)\ dy\, dx$$

$$= \frac{3}{4}\int_{x=0}^{x=2} \int_{y=0}^{y=2-x} (2x - x^2 - xy)\, dy\, dx$$

$$= \frac{3}{4}\int_{x=0}^{x=2} (2xy - x^2 y - \frac{xy^2}{2})\Big|_{y=0}^{y=2-x} dx$$

$$= \frac{3}{4}\int_{x=0}^{x=2} [2x(2-x) - x^2(2-x) - \frac{x(2-x)^2}{2}\ ]dx$$

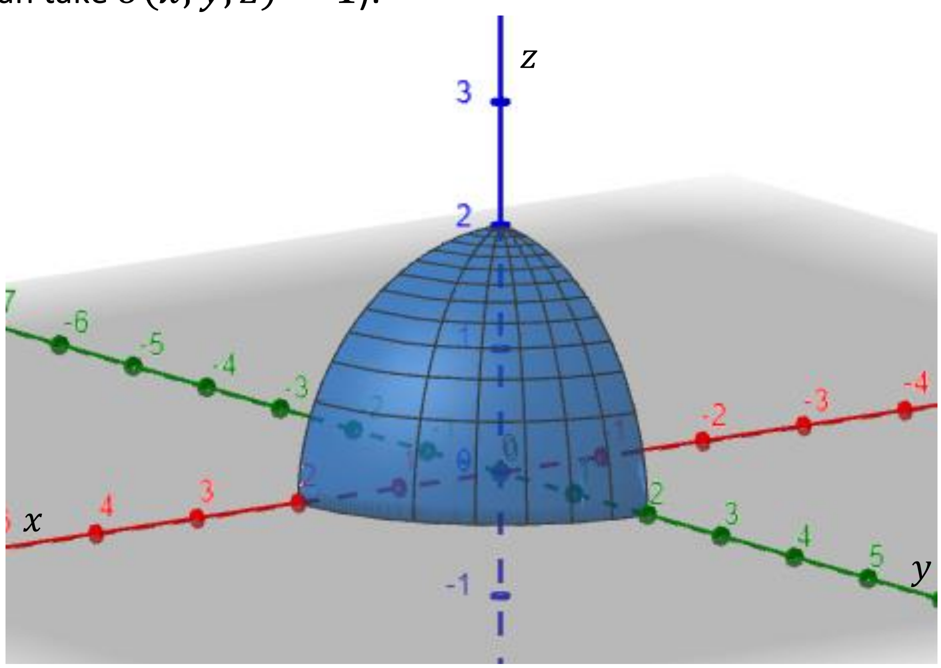
$$= \frac{3}{4}\int_{x=0}^{x=2} [x(2-x)^2 - \frac{x(2-x)^2}{2}]\, dx = \frac{3}{8}\int_{x=0}^{x=2} x(2-x)^2\, dx$$

$$= \frac{3}{8}\int_{x=0}^{x=2} x(4-4x+x^2)\ dx = \frac{3}{8}\int_{x=0}^{x=2} (4x-4x^2+x^3)dx = \frac{1}{2}.$$

Since the problem is symmetrical in x, y, and z:

$$\bar{x}, \bar{y}, \bar{z} = \frac{1}{2}.$$

Ex. Find an expression in spherical coordinates for the center of mass of the region bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the planes: $z = 0, y = 0$, and $x = 0$, and $x \geq 0,\ \ y \geq 0$, and $z \geq 0$ if the density is uniform (i.e. we can take $\delta(x, y, z) = 1$).



In spherical coordinates the region is:

$$0 \leq \rho \leq 2, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad \text{and } 0 \leq \theta \leq \frac{\pi}{2}.$$

$$\text{mass} = m = \iiint_W \delta(x,y,z)\,dz\,dy\,dx = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$\bar{x} = \frac{1}{m} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho \sin\phi \cos\theta)(\rho^2 \sin\phi)\,d\rho\,d\phi\,d\theta$$

$$\bar{x} = \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho^3 \sin^2\phi \cos\theta)\,d\rho\,d\phi\,d\theta}{m}$$

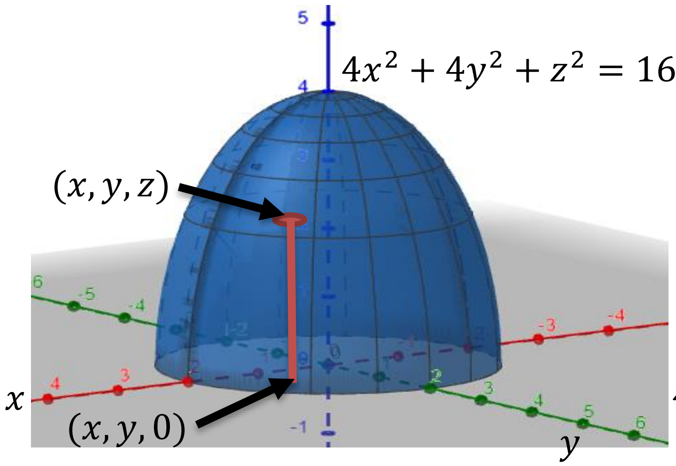
$$\bar{y} = \frac{1}{m} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho \sin\phi \sin\theta)(\rho^2 \sin\phi)\,d\rho\,d\phi\,d\theta$$

$$\bar{y} = \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho^3 \sin^2\phi \sin\theta)\,d\rho\,d\phi\,d\theta}{m}$$

$$\bar{z} = \frac{1}{m} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho \cos\phi)(\rho^2 \sin\phi)\,d\rho\,d\phi\,d\theta$$

$$\bar{z} = \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=2} (\rho^3 \cos\phi \sin\phi)\,d\rho\,d\phi\,d\theta}{m}\,.$$

Ex. Find the mass of an ellipsoidal solid, W, bounded by:
$4x^2 + 4y^2 + z^2 = 16$ and $z = 0$; with $z \geq 0$, with the density at a point is proportional to the distance to the xy plane.



$$\rho(x, y, z) = kz$$

Using cylindrical coordinates

$$z = \sqrt{16 - 4x^2 - 4y^2} = \sqrt{16 - 4r^2}$$

$$\text{mass} = \iiint_W \rho(x, y, z)\, dz\, dy\, dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=\sqrt{16-4r^2}} (kzr)\, dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \frac{kz^2 r}{2} \Bigg|_0^{\sqrt{16-4r^2}} dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \frac{k}{2}(16 - 4r^2) r\, dr\, d\theta$$

$$= 2k \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4r - r^3)\, dr\, d\theta = 2k \int_{\theta=0}^{\theta=2\pi} (2r^2 - \frac{r^4}{4})\Big|_0^2\, d\theta$$

$$= 2k \int_{\theta=0}^{\theta=2\pi} (8 - 4)\ d\theta = 2k(4\theta)|_0^{2\pi} = 16k\pi\ .$$

<u>Moments of Inertia</u>

The moment of inertia measures a body's response to efforts to rotate it about a line. For example, it is harder to rotate a large plate than a small plate of the same mass.

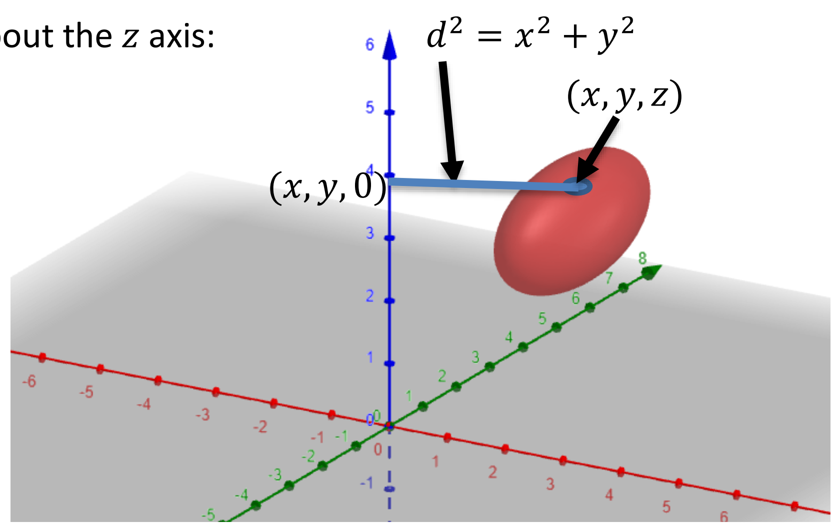
Moments of inertia about the coordinate axes:

$$I_x = \iiint_W (y^2 + z^2)\delta \, dx \, dy \, dz$$

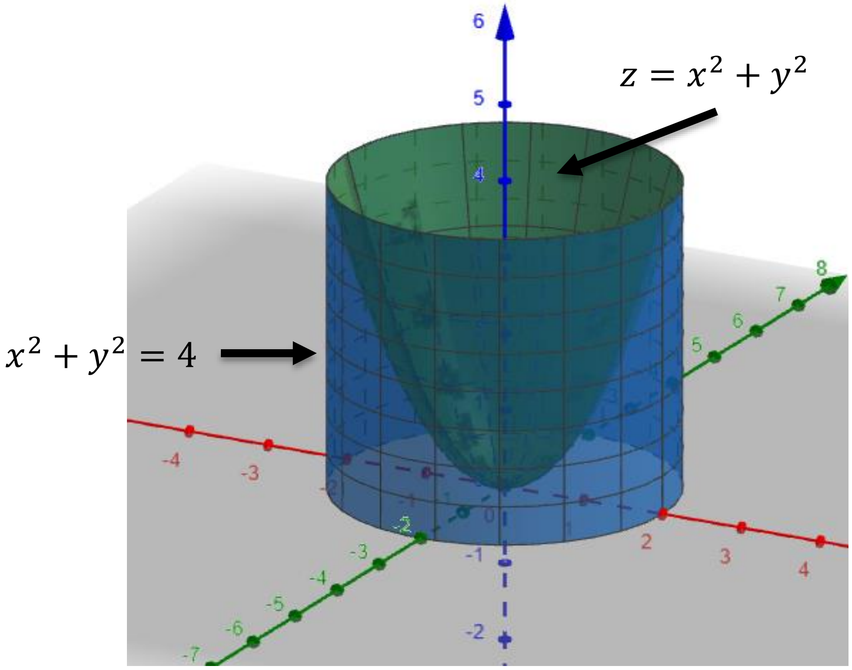
$$I_y = \iiint_W (x^2 + z^2)\delta \, dx \, dy \, dz$$

$$I_z = \iiint_W (x^2 + y^2)\delta \, dx \, dy \, dz$$

Inertia about the z axis:



Ex. Find the moment of inertia, I_z, about the z axis of the solid bounded above by the paraboloid: $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$, and the xy-plane assuming the mass density, δ, is constant.



$$I_z = \iiint_W (x^2 + y^2)\delta \; dx \; dy \; dz$$

The paraboloid and cylinder intersect at $z = 4$.

$$I_z = \iint_{x^2+y^2 \le 4} \int_{z=0}^{z=x^2+y^2} (x^2 + y^2) \, \delta \; dx \; dy \; dz$$

This is easiest to do if we use cylindrical coordinates:

$$I_z = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=r^2} \delta(r^2)\, r\, dz\, dr\, d\theta$$

$$= \delta \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=r^2} r^3\, dz\, dr\, d\theta = \delta \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^3 z \Bigg|_{z=0}^{z=r^2} dr\, d\theta$$

$$= \delta \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^5\ dr\, d\theta = \delta \int_{\theta=0}^{\theta=2\pi} \frac{r^6}{6}\Bigg|_{r=0}^{r=2} d\theta$$

$$= \delta \int_{\theta=0}^{\theta=2\pi} \frac{2^6}{6}\, d\theta = \ \delta \left(\frac{2^6}{6}\right) \theta \Bigg|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{64\pi\delta}{3}.$$

Ex. Set up, but do not evaluate, the moments of inertia, I_x, I_y in the previous example in cylindrical coordinates.

$$I_x = \iiint_W (y^2 + z^2)\delta \, dx \, dy \, dz$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=r^2} \delta(r^2 \sin^2 \theta + z^2) \, r \, dz \, dr \, d\theta \, .$$

$$I_y = \iiint_W (x^2 + z^2)\delta \, dx \, dy \, dz$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=r^2} \delta(r^2 \cos^2 \theta + z^2) \, r \, dz \, dr \, d\theta \, .$$