The Change of Variables Theorem

In one variable calculus, we sometimes need to change the variable to compute an integral.

By letting $x = g(u)$ and $dx = g'(u)\, du$ we can change:

$$\int_{x=a}^{x=b} f(x)\, dx = \int_{u=c}^{u=d} f\big(g(u)\big)\, g'(u)\, du$$

where $a = g(c)$ and $b = g(d)$.

Ex. Evaluate $\int_0^1 \sqrt{1-x^2}\, dx$

Let: $x = \sin u$

$dx = \cos u \; du$

$x = 0 = \sin u \qquad \Rightarrow u = 0$

$x = 1 = \sin u \qquad \Rightarrow u = \frac{\pi}{2}$

$$\int_{x=0}^{x=1} \sqrt{1-x^2}\, dx = \int_{u=0}^{u=\frac{\pi}{2}} \left(\sqrt{1-\sin^2 u}\right) \cos u\, du$$

$$= \int_{u=0}^{u=\frac{\pi}{2}} \cos^2 u\, du = \int_{u=0}^{u=\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right) du$$

$$= \frac{1}{2}u + \frac{1}{4}\sin 2u\bigg|_0^{\frac{\pi}{2}} = \left(\frac{\pi}{4} + 0\right) - (0+0) = \frac{\pi}{4}.$$

If we have $\iint_R f(x,y)dA$, how do we change variables in the integral when $x = g(u,v)$ and $y = h(u,v)$?

Def. (Jacobian Determinant): Let $T: D^* \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable mapping where $T(u,v) = \big(g(u,v), h(u,v)\big)$. That is, $x = g(u,v), y = h(u,v)$. The **Jacobian Determinant of T**, is written and defined by the following:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Ex. Let T be the change of coordinates given by $x = r\cos\theta$ and $y = r\sin\theta$ (i.e. $T(r,\theta) = (r\cos\theta, r\sin\theta)$). Calculate $\frac{\partial(x,y)}{\partial(r,\theta)}$.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r.$$

In one variable calculus, when we changed variables by letting $x = g(u)$ we got:

$$\int_{x=a}^{x=b} f(x)\, dx = \int_{u=c}^{u=d} f\big(g(u)\big)\, g'(u)\, du.$$

So $g'(u)$ was the "correction" factor we needed in the integral after substituting $x = g(u)$ into the integral (we also needed to adjust the endpoints of integration by $a = g(c)$ and $b = g(d)$).

For a double integral, when we substitute $x = g(u, v)$ and $y = h(u, v)$ the "adjustment factor" in the integral is:

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right|$$

So the absolute value of the Jacobian determinant plays the role of $g'(u)$ for a double integral (this is also true for triple integrals).

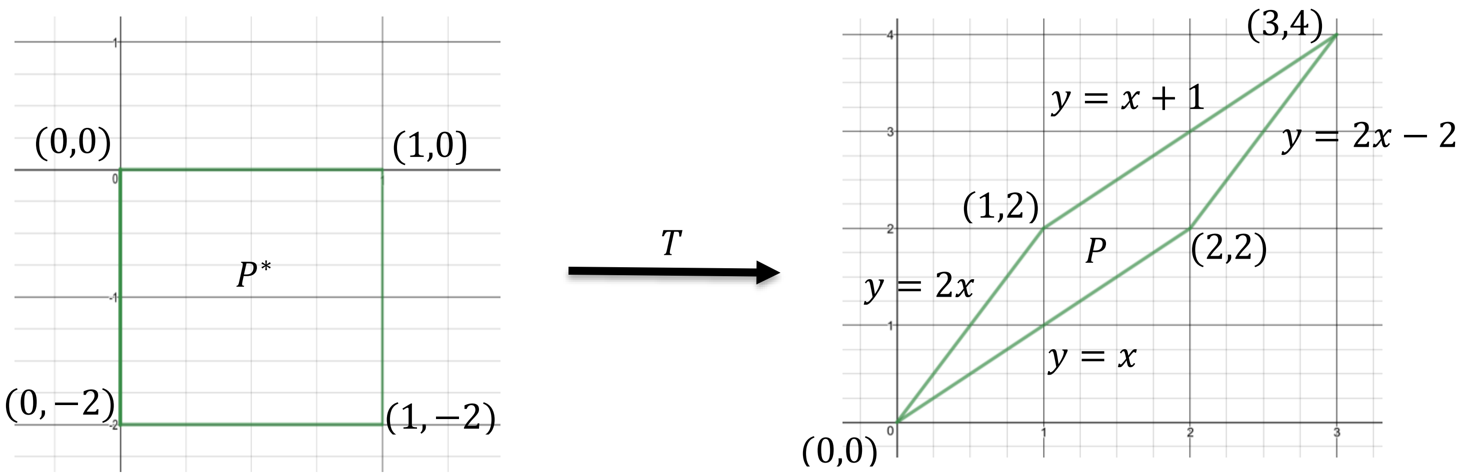
If we substitute $x = g(u, v), y = h(u, v)$ in a double integral, the change of variable formula is:

$$\boldsymbol{\iint\limits_{R} f(x, y) dA = \iint\limits_{S} f\big(g(u, v), h(u, v)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\, du\, dv.}$$

As with a one variable integral, we will also have to adjust the endpoints of integration to reflect the new variables. In this case, if $T(u, v) = \big(g(u, v), h(u, v)\big)$, then S is the set where $T(S) = R$.

Ex. Let P be the parallelogram bounded by $y = 2x, y = 2x - 2, y = x$, and $y = x + 1$. Evaluate $\iint_P (x^2 - y^2)dx\,dy$ by making the change of variables: $x = u - v, \quad y = 2u - v.$

The transformation $T(u, v) = (u - v, 2u - v)$ maps the rectangle, P^*, bounded by $v = 0, v = -2, u = 0, u = 1$ into the parallelogram P.



We can see this by asking what set in the uv plane gets mapped to each of the sides of P.

$$
\begin{aligned}
&y = 2x: && 2u - v = 2(u - v) && \Rightarrow v = 0\\
&y = 2x - 2: && 2u - v = 2(u - v) - 2 && \Rightarrow v = -2\\
&y = x: && 2u - v = u - v && \Rightarrow u = 0\\
&y = x + 1: && 2u - v = (u - v) + 1 && \Rightarrow u = 1
\end{aligned}
$$

Now find $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

$$x = u - v \qquad \frac{\partial x}{\partial u} = 1 \quad ; \quad \frac{\partial x}{\partial v} = -1$$

$$y = 2u - v \qquad \frac{\partial y}{\partial u} = 2 \quad ; \quad \frac{\partial x}{\partial v} = -1$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\right| = 1.$$

So now we can write:

$$\iint_P x^2 - y^2\, dx\, dy = \iint_{P^*} (u-v)^2 - (2u-v)^2\, du\, dv$$

$$= \int_{v=-2}^{v=0} \int_{u=0}^{u=1} (-3u^2 + 2uv) du\, dv = \int_{v=-2}^{v=0} -u^3 + \frac{2u^2 v}{2}\Bigg|_{u=0}^{u=1} dv$$

$$= \int_{v=-2}^{v=0} (-1+v)\ dv = -v + \frac{v^2}{2}\Bigg|_{v=-2}^{v=0}$$

$$= 0 - (2+2) = -4\,.$$

Ex. Suppose we want to find the area of the unit disk, $D = x^2 + y^2 \leq 1$, using a double integral.

Let's change to polar coordinates. That is $T(r,\theta) = (r\cos\theta\,, r\sin\theta)$).
So $\quad x = rcos\theta, \quad y = rsin\theta.$

$$A(D) = \iint_D dx\, dy = \iint_{[0,1]\times[0,2\pi]} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$$

from an earlier example we found $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$, so we have:

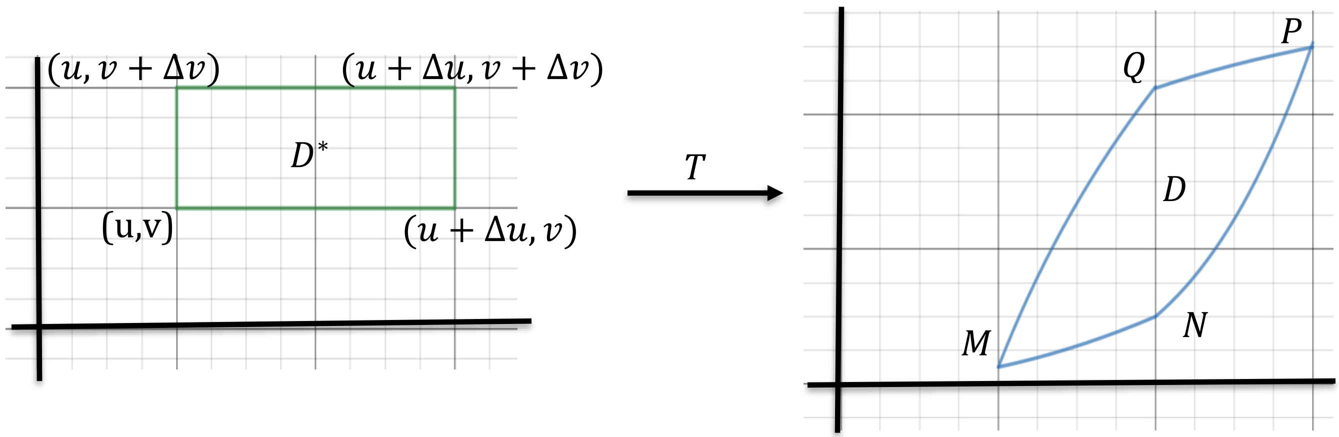
$$= \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2} r^2|_0^1\ d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi.$$

Why does this change of variable formula work?

Suppose we have $T: D^* \to D$ by $T(u,v) = \big(g(u,v), h(u,v)\big)$.

In other words, we are changing variables by letting

$x = g(u,v)$ and $y = h(u,v)$.



where:

$$M = \big(g(u,v), h(u,v)\big)$$
$$N = \big(g(u+\Delta u, v), h(u+\Delta u, v)\big)$$
$$P = \big(g(u+\Delta u, v+\Delta v), h(u+\Delta u, v+\Delta v)\big)$$
$$Q = \big(g(u, v+\Delta v), h(u, v+\Delta v)\big)$$

$A(D) \approx \left\| \overrightarrow{MN} \times \overrightarrow{MQ} \right\|$ (D is approximately a parallelogram)

$$g_u(u,v) \approx \frac{g(u+\Delta u,v) - g(u,v)}{\Delta u}$$

$$h_u(u,v) \approx \frac{h(u+\Delta u,v) - h(u,v)}{\Delta u}.$$

$$\overrightarrow{MN} = [g(u+\Delta u, v) - g(u,v)]\vec{\imath} + [h(u+\Delta u, v) - h(u,v)]\vec{\jmath}$$

$$\approx [g_u(u,v)\Delta u]\vec{\imath} + [h_u(u,v)\Delta u]\vec{\jmath}$$

$$= \frac{\partial x}{\partial u}(\Delta u)\vec{\imath} + \frac{\partial y}{\partial u}(\Delta u)\vec{\jmath}$$

Similarly:

$$\overrightarrow{MQ} \approx \frac{\partial x}{\partial v}(\Delta v)\vec{\imath} + \frac{\partial y}{\partial v}(\Delta v)\vec{\jmath}$$

$$\overrightarrow{MN} \times \overrightarrow{MQ} \approx \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial x}{\partial u}(\Delta u) & \frac{\partial y}{\partial u}(\Delta u) & 0 \\ \frac{\partial x}{\partial v}(\Delta v) & \frac{\partial y}{\partial v}(\Delta v) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} (\Delta u)(\Delta v)\vec{k}$$

$$A(D) \approx \left|\overrightarrow{MN} \times \overrightarrow{MQ}\right| \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| (\Delta u)(\Delta v)$$

So as $\Delta u, \Delta v \to 0$:

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du \, dv$$

In particular, when changing to polar coordinates:

$$\iint_D f(x,y)\, dx\, dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

i.e. $dA = dx\, dy = r\, dr\, d\theta$.

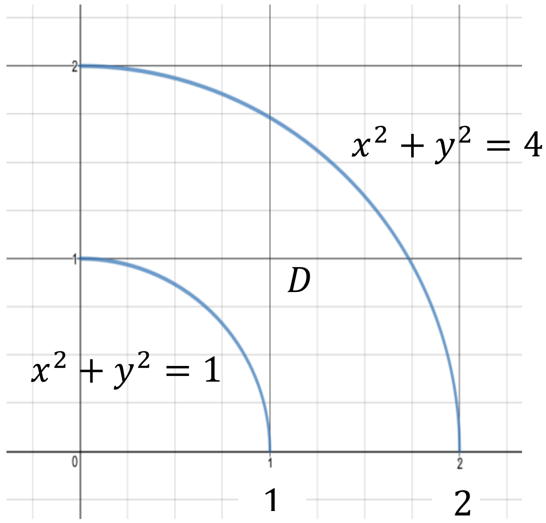
Ex. Evaluate $\iint_D (x+y)\,dx\,dy$ where D is the region in the first quadrant lying between $x^2+y^2=1$ and $x^2+y^2=4$.

When integrating over a region in the plane that is a disk, an annulus, or a portion of a disk or annulus, changing to polar coordinates is often useful.

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{\partial(x,y)}{\partial(r,\theta)} = r$



$$\iint_D (x+y)\,dx\,dy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

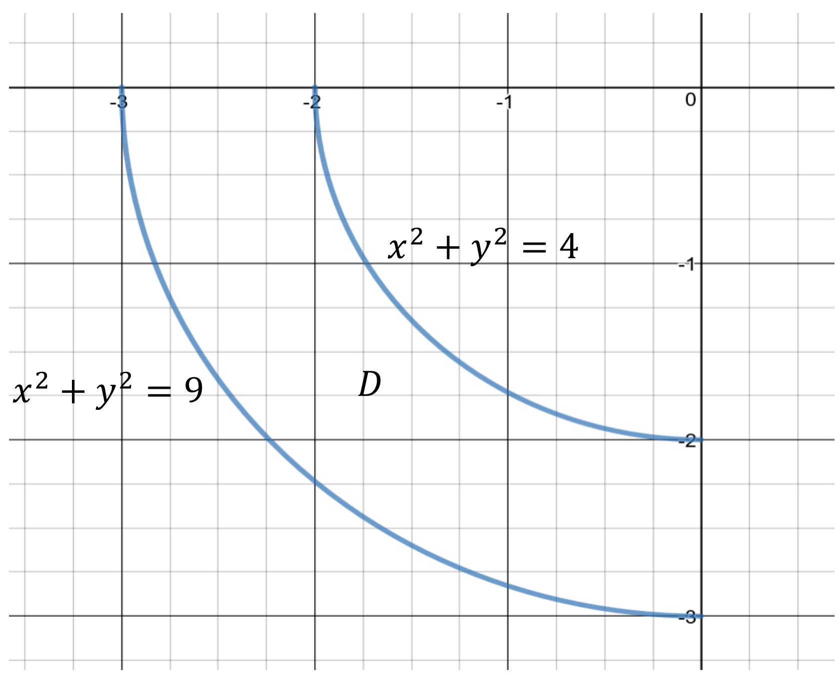
$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} (r^2\cos\theta + r^2\sin\theta)\,dr\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\frac{r^3}{3}\cos\theta + \frac{r^3}{3}\sin\theta)\right|_{r=1}^{r=2} d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} [\left(\frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta\right) - \left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta\right)]d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{7}{3}(\cos\theta + \sin\theta)\, d\theta = \frac{7}{3}(\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{7}{3}\big((1-0)-(0-1)\big) = \frac{14}{3}.$$

Ex. Evaluate $\iint_D \sin(x^2+y^2)\, dx\, dy$ where D is the region bounded by $x^2+y^2=4$ and $x^2+y^2=9$, when $x \le 0$ and $y \le 0$.



Changing to polar coordinates we get:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

$$\iint_D \sin(x^2+y^2)\,dx\,dy = \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} \int_{r=2}^{r=3} (\sin(r^2))r\,dr\,d\theta$$

Let $u = r^2;$ when $r = 2,\ u = 4.$

$du = 2rdr;$ when $r = 3,\ u = 9.$

$\frac{1}{2}du = rdr.$

$$\int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} \int_{r=2}^{r=3} (\sin(r^2))r\,dr\,d\theta = \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} \int_{u=4}^{u=9} \frac{1}{2}(\sin(u))\,du\,d\theta$$

$$= \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} -\frac{1}{2}cosu\Big|_{u=4}^{u=9} d\theta$$

$$= \int_{\theta=\pi}^{\theta=\frac{3\pi}{2}} -\frac{1}{2}(cos9 - cos4)d\theta$$

$$= -\frac{1}{2}(cos9 - cos4)\theta\Big|_{\theta=\pi}^{\theta=\frac{3\pi}{2}}$$

$$= \frac{\pi}{4}(cos4 - cos9).$$

Ex. Show that:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Let: $\quad I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

Then:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{-x^2}\cdot e^{-y^2})\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\,dx\,dy$$

Now change to polar coordinates:

$$x^2 + y^2 = r^2$$
$$dx\,dy = r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} \left(e^{-r^2}\right)r\,dr\,d\theta = \lim_{\alpha\to\infty}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\alpha} \left(e^{-r^2}\right)r\,dr\,d\theta$$

$$= \lim_{\alpha\to\infty}\int_{\theta=0}^{\theta=2\pi} -\frac{e^{-r^2}}{2}\Bigg|_{r=0}^{r=\alpha} d\theta = \lim_{\alpha\to\infty}\int_{\theta=0}^{\theta=2\pi}\left(\frac{-e^{-\alpha^2}+1}{2}\right)d\theta$$

$$= \lim_{\alpha \to \infty} \left(\frac{-e^{-\alpha^2} + 1}{2} \right) \theta \Big|_0^{2\pi} = \lim_{\alpha \to \infty} \pi\left(-e^{-\alpha^2} + 1\right) = \pi.$$

So $I^2 = \pi \Rightarrow I = \sqrt{\pi}$ (since $I \geq 0$)

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

<u>Change of variables for triple integrals</u>

Let $T: W \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable defined by:

$$T(u, v, w) = \big(g(u, v, w), h(u, v, w), k(u, v, w)\big)$$

i.e. $x = g(u, v, w)$, $y = h(u, v, w)$, $z = k(u, v, w)$

Then the Jacobian determinant of T is:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Change of variables formula for triple integrals:

$$\iiint_{\boldsymbol{W}} \boldsymbol{f(x,y,z)\,dx\,dy\,dz}$$

$$= \iiint_{\boldsymbol{W^*}} \boldsymbol{f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)} \left|\frac{\boldsymbol{\partial(x,y,z)}}{\boldsymbol{\partial(u,v,w)}}\right| \boldsymbol{du\,dv\,dw}$$

Where W^* is the region in uvw space corresponding to W in xyz space

Cylindrical coordinates:

$$x = r\cos\theta\,; \qquad \frac{\partial x}{\partial r} = cos\theta \qquad \frac{\partial x}{\partial \theta} = -rsin\theta \qquad \frac{\partial x}{\partial z} = 0$$

$$y = r\sin\theta \qquad \frac{\partial y}{\partial r} = sin\theta \qquad \frac{\partial y}{\partial \theta} = rcos\theta \qquad \frac{\partial y}{\partial z} = 0$$

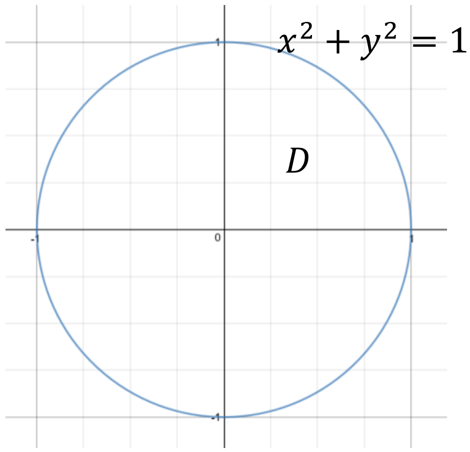
$$z = z \qquad \frac{\partial z}{\partial r} = 0 \qquad \frac{\partial z}{\partial \theta} = 0 \qquad \frac{\partial z}{\partial z} = 1$$

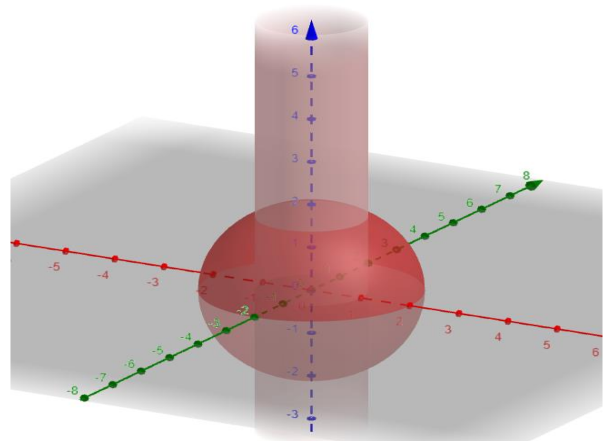
$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Change of variables formula – cylindrical coordinates:

$$\iiint_{W} f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta\,, r\sin\theta\,, z)\,r\,dr\,d\theta\,dz.$$

Ex. Find the volume of the solid that lies within the cylinder, $x^2 + y^2 = 1$, and the sphere, $x^2 + y^2 + z^2 = 4$.





$$V = \iiint\limits_W dx\, dy\, dz \iint_{x^2+y^2\le 1} [\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz]dydx$$

Changing to cylindrical coordinates:

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} zr \Big|_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} drd\theta$$

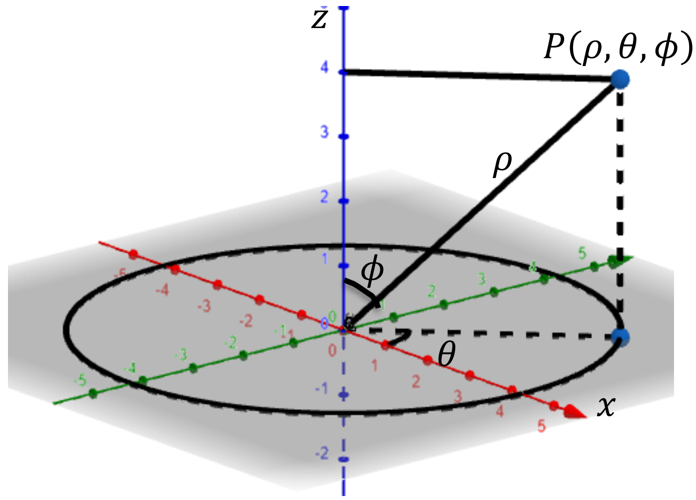
$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 2r(4-r^2)^{\frac{1}{2}}\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} -\frac{2}{3}(4-r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=1} d\theta$$

$$= -\frac{2}{3}\int_{\theta=0}^{\theta=2\pi} \left(3^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) d\theta = -\frac{2}{3}\int_{\theta=0}^{\theta=2\pi} \left(3^{\frac{3}{2}} - 8\right) d\theta$$

$$= -\frac{2}{3}\left(3^{\frac{3}{2}} - 8\right)\theta\Big|_0^{2\pi} = \frac{2}{3}(8-3\sqrt{3})2\pi = \frac{4\pi}{3}(8-3\sqrt{3}).$$

Spherical coordinates

Recall that any point in $\mathbb{R}^3$ can be thought of as residing on some sphere with center $(0,0,0)$.



$P(\rho,\theta,\phi) = (x,y,z)$
$\rho = |OP|\ ;\ \rho \geq 0$
$x = \rho \sin\phi\,(\cos\theta)$
$y = \rho \sin\phi\,(\sin\theta)$
$z = \rho\cos\phi$
where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Now let's calculate the Jacobian for changing from Cartesian coordinates to spherical coordinates:

$$\frac{\partial x}{\partial \rho} = (\sin\phi)\,(\cos\theta) \qquad \frac{\partial x}{\partial \theta} = -\rho\,(\sin\phi)\,(\sin\theta) \qquad \frac{\partial x}{\partial \phi} = \rho\,(\cos\phi)(\cos\theta)$$

$$\frac{\partial y}{\partial \rho} = (\sin\phi)\,(\sin\theta) \qquad \frac{\partial y}{\partial \theta} = \rho\,(\sin\phi)\,(\cos\theta) \qquad \frac{\partial y}{\partial \phi} = \rho\,(\cos\phi)(\sin\theta)$$

$$\frac{\partial z}{\partial \rho} = \cos\phi \qquad \frac{\partial z}{\partial \theta} = 0 \qquad \frac{\partial z}{\partial \phi} = -\rho\,(\sin\phi)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} (\sin\phi)\,(\cos\theta) & -\rho\,(\sin\phi)\,(\sin\theta) & \rho\,(\cos\phi)(\cos\theta) \\ (\sin\phi)\,(\sin\theta) & \rho\,(\sin\phi)\,(\cos\theta) & \rho\,(\cos\phi)(\sin\theta) \\ \cos\phi & 0 & -\rho\,(\sin\phi) \end{vmatrix}$$

$$\begin{aligned} &= (\sin\phi)\,(\cos\theta)\,(-\rho^2 \sin^2\phi\, cos\theta) \\ &\quad + \rho\,(\sin\phi)\,(\sin\theta)\,(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta) \\ &\quad + \rho(\cos\phi)(\cos\theta)(-\rho\,(\sin\phi)\,\big((\cos\theta)\,(\cos\phi)\big) \end{aligned}$$

$$= -\rho^2 \sin^3 \phi\,(\cos^2 \theta) + \rho\,(\sin \phi)\,(\sin \theta)(-\rho \sin \theta)$$
$$- \rho^2(\cos^2 \phi)(\sin \phi)(\cos^2 \theta)$$

$$= -\rho^2 \sin^3 \phi\,(\cos^2 \theta) - \rho^2\,(\sin \phi)\,(\sin^2 \theta) - \rho^2\,(\cos^2 \phi)\,(\sin \phi) \cos^2 \theta$$

$$= -\rho^2(\sin \phi)(\cos^2 \theta)[\sin^2 \phi + \cos^2 \phi] - \rho^2(\sin \phi)(\sin^2 \theta)$$

$$= -\rho^2(\sin \phi)[\cos^2 \theta + \sin^2 \theta] = -\rho^2(\sin \phi).$$

So: $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi\,)}\right| = \rho^2(\sin \phi)$

Change of variable formula for spherical coordinates:

$$\iiint\limits_{W} f(x, y, z)\; dx\; dy\; dz$$

$$= \iiint\limits_{W^*} f(\rho \sin \phi\,(\cos \theta), \rho \sin \phi\,(\sin \theta), \rho \cos \phi)\, \rho^2 \sin \phi\;\; d\rho\; d\theta\; d\phi$$

Ex. Evaluate $\iiint_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV$ using spherical coordinates, where

a) W is the set $\{(x,y,z) \in \mathbb{R}^3 |\ x^2+y^2+z^2 \le 1,\ z \ge 0\}$

b) W is the solid bounded by $x^2+y^2+z^2=4,\ x^2+y^2+z^2=1,$ and $z=0$, where $z \ge 0$.

a) $x^2+y^2+z^2 \le 1,\ z \ge 0$ in $\mathbb{R}^3$ in spherical coordinates is the set:

$$0 \le \rho \le 1, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \frac{\pi}{2}$$

$$\iiint\limits_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV = \int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} e^{(\rho^2)^{\frac{3}{2}}}(\rho^2)\sin\phi\ d\rho\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} (e^{\rho^3})(\rho^2)\sin\phi\ d\rho\, d\theta\, d\phi$$

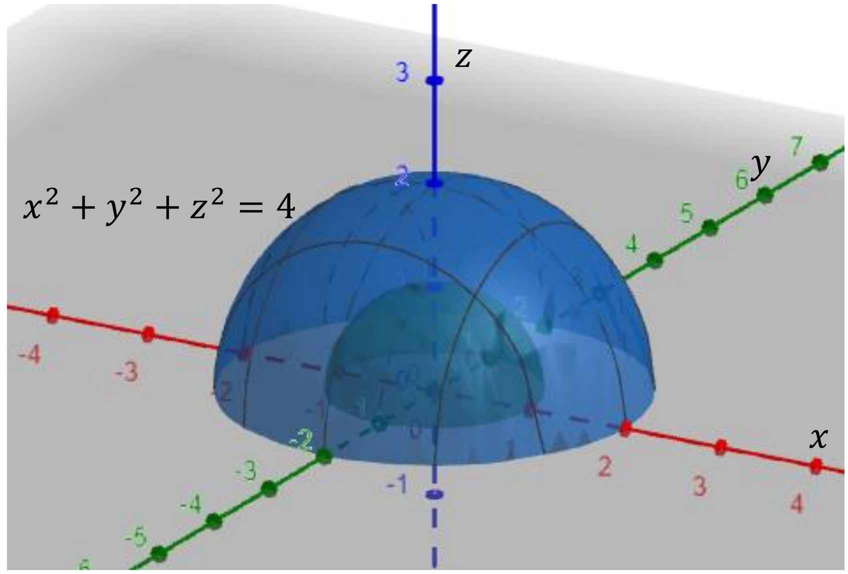
$$= \int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=2\pi} \frac{1}{3}e^{\rho^3}\sin\phi\Big|_{\rho=0}^{\rho=1}\ d\theta\, d\phi = \int_{\phi=0}^{\phi=\pi/2}\int_{\theta=0}^{\theta=2\pi} \frac{1}{3}(e-1)\sin\phi\ d\theta\, d\phi$$

$$= \int_{\phi=0}^{\phi=\pi/2} \frac{1}{3}(e-1)(\sin\phi)\theta\Big|_{\theta=0}^{\theta=2\pi}\ d\phi = \int_{\phi=0}^{\phi=\pi/2} \frac{2\pi}{3}(e-1)\sin\phi\, d\phi$$

$$= \frac{2\pi}{3}(e-1)(-\cos\phi)\Big|_{\phi=0}^{\phi=\pi/2} = \frac{2\pi}{3}(e-1)\big(0-(-1)\big) = \frac{2\pi}{3}(e-1).$$

b) W, the solid bounded by $x^2+y^2+z^2=4$, $x^2+y^2+z^2=1$, and $z=0$, where $z\geq 0$.

$1\leq \rho \leq 2$
$0\leq \phi \leq \frac{\pi}{2}$
$0\leq \theta \leq 2\pi.$



$$\iiint\limits_W e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV=\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=1}^{\rho=2}(e^{\rho^3})(\rho^2)\sin\phi\ \,d\rho\,d\theta\,d\phi$$

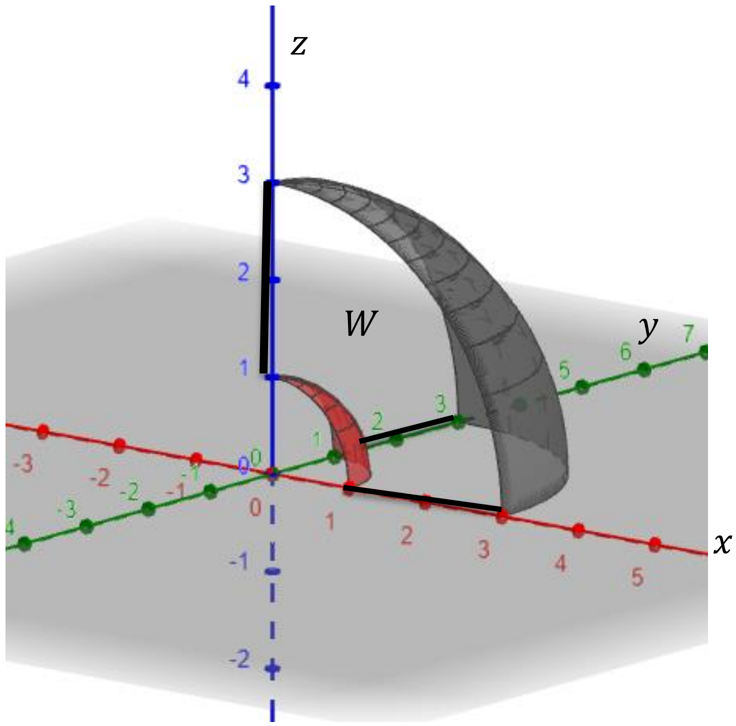
$$=\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}\left(\frac{1}{3}e^{\rho^3}\sin\phi\right)\Bigg|_{\rho=1}^{\rho=2}d\theta\,d\phi=\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}\frac{1}{3}(e^8-e)\sin\phi\,d\theta\,d\phi$$

$$=\frac{1}{3}(e^8-e)\int_{\phi=0}^{\phi=\frac{\pi}{2}}(\sin\phi)\,\theta\Bigg|_{\theta=0}^{\theta=2\pi}d\phi=\frac{2\pi}{3}(e^8-e)\int_{\phi=0}^{\phi=\frac{\pi}{2}}\sin\phi\ \,d\phi$$

$$=\frac{2\pi}{3}(e^8-e)(-\cos\phi)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}$$
$$=\frac{2\pi}{3}(e^8-e).$$

Ex. Set up the integral $\iiint_W \frac{dxdydz}{(x^2+y^2+z^2)^2}\, dV$ in spherical coordinates if

W is the set of points in the first octant ($x > 0, y > 0, z > 0$)
that are bounded by $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$.



In spherical coordinates, the set of points in $\mathbb{R}^3$ that are bounded by $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$ are represented by:

$$1 \leq \rho \leq 3, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

In spherical coordinates, the set of points in $\mathbb{R}^3$ that are in the first octant are represented by:

$$0 < \rho, \quad 0 < \phi < \frac{\pi}{2}, \quad 0 < \theta < \frac{\pi}{2}.$$

Thus the set W is the set where:

$$1 \leq \rho \leq 3, \quad 0 < \phi < \frac{\pi}{2}, \quad 0 < \theta < \frac{\pi}{2}.$$

In spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 \quad \text{and}$$

$$dV = dxdydz = \rho^2 \sin\phi \ d\rho \, d\theta \, d\phi$$

So we have:

$$\iiint_W \frac{dxdydz}{\left(x^2+y^2+z^2\right)^2} \ dV = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=1}^{\rho=3} \left(\frac{1}{\rho^4}\right)(\rho^2) \sin\phi \ d\rho \, d\theta \, d\phi.$$