Changing Variables in Subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$

Let D^* be a subset of $\mathbb{R}^2$ then $T: D^* \to \mathbb{R}^2$ is called a **change of variables**. Usually we will be interested in the case where T is a continuously differentiable function. The **image** of D^* under T, $T(D^*)$, is the set of points:

$$(x, y) = T(x^*, y^*) \text{ for } (x^*, y^*) \in D^*$$

Ex. Let $D^* \subseteq \mathbb{R}^2$ be the rectangle $D^* = [0,1] \times [0, 2\pi]$. So,

$$D^* = \{(r, \theta) \in \mathbb{R}^2 | \, 0 \le r \le 1, \;\; 0 \le \theta \le 2\pi\}$$

Let $T: D^* \to \mathbb{R}^2$ by $T(r, \theta) = (r\cos\theta, r\sin\theta)$.
Find $T(D^*)$.

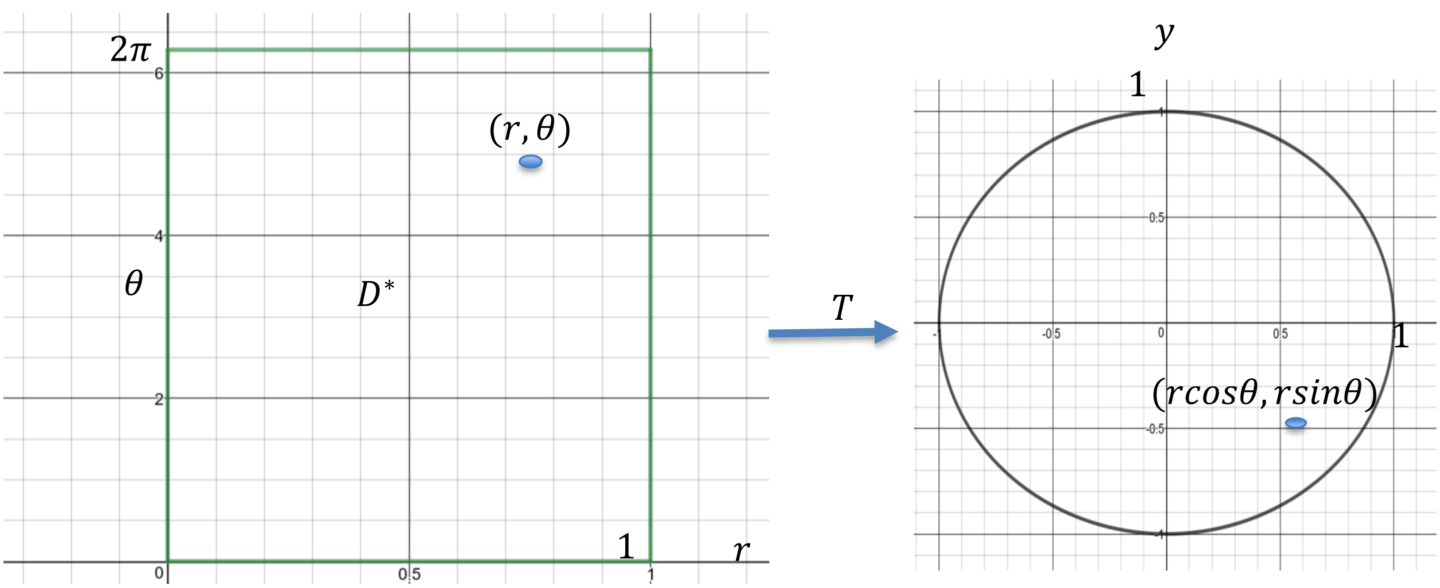
All of the points of $T(D^*)$ look like $(r\cos\theta, r\sin\theta)$, where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
If we let $x = r\cos\theta$ and $y = r\sin\theta$, then:

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 1$$

So every point (x, y) in $T(D^*)$ must have $x^2 + y^2 \le 1$, thus:

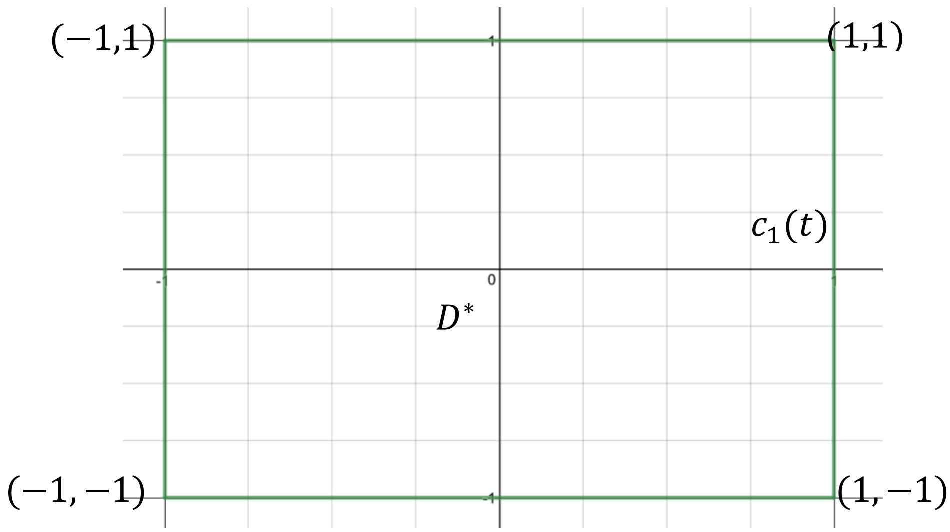
$$T(D^*) \subseteq \text{the unit disk}$$

But any point in the unit disk can be written as $(r\cos\theta, r\sin\theta)$ for some $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
Thus, $T(D^*)$ is the unit disk.



Ex. Let $D^* = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$, a square with side length of 2 centered at the origin. Let $T(x, y) = \left(\frac{(x-y)}{2}, \frac{(x+y)}{2}\right)$. Find $T(D^*)$.

First, let's see what T does to the boundary of D^*.



$c_1(t) = < 1, t > \; ; \; -1 \le t \le 1$

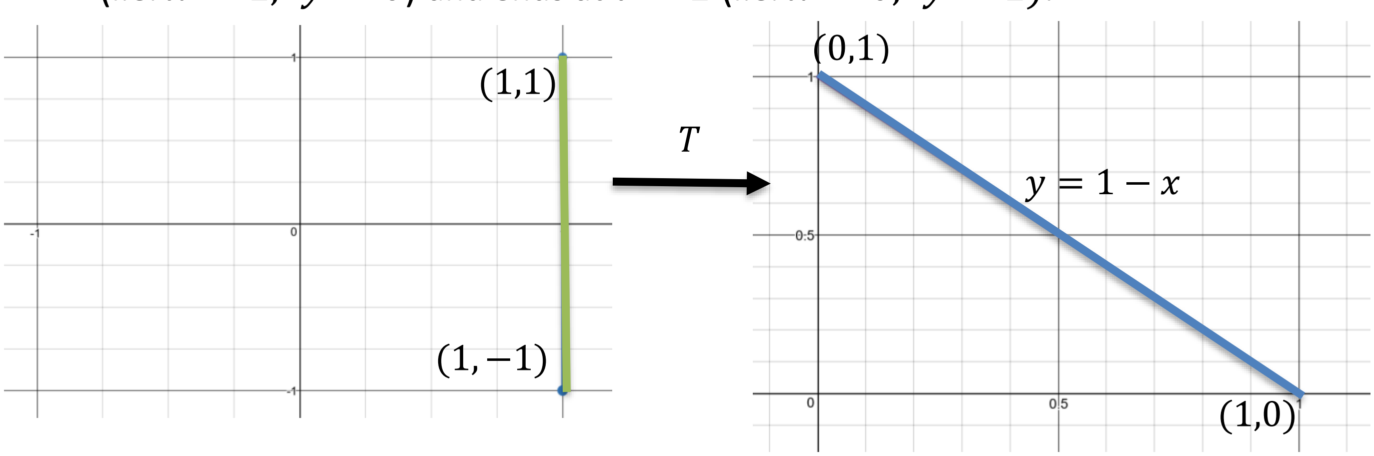
$T(c_1(t)) = \left(\frac{1-t}{2}, \frac{1+t}{2}\right) \; ; \; -1 \le t \le 1$

So $x = \frac{1-t}{2}$ and $y = \frac{1+t}{2}$

We can eliminate the t by adding the equations to get:

$$x + y = 1$$

Since $-1 \le t \le 1$, we get the portion of the line that starts at $t = -1$ (i.e. $x = 1,\ y = 0$) and ends at $t = 1$ (i.e. $x = 0,\ y = 1$).



Similarly:

$c_2(t) = < t, 1 > \qquad -1 \leq t \leq 1$

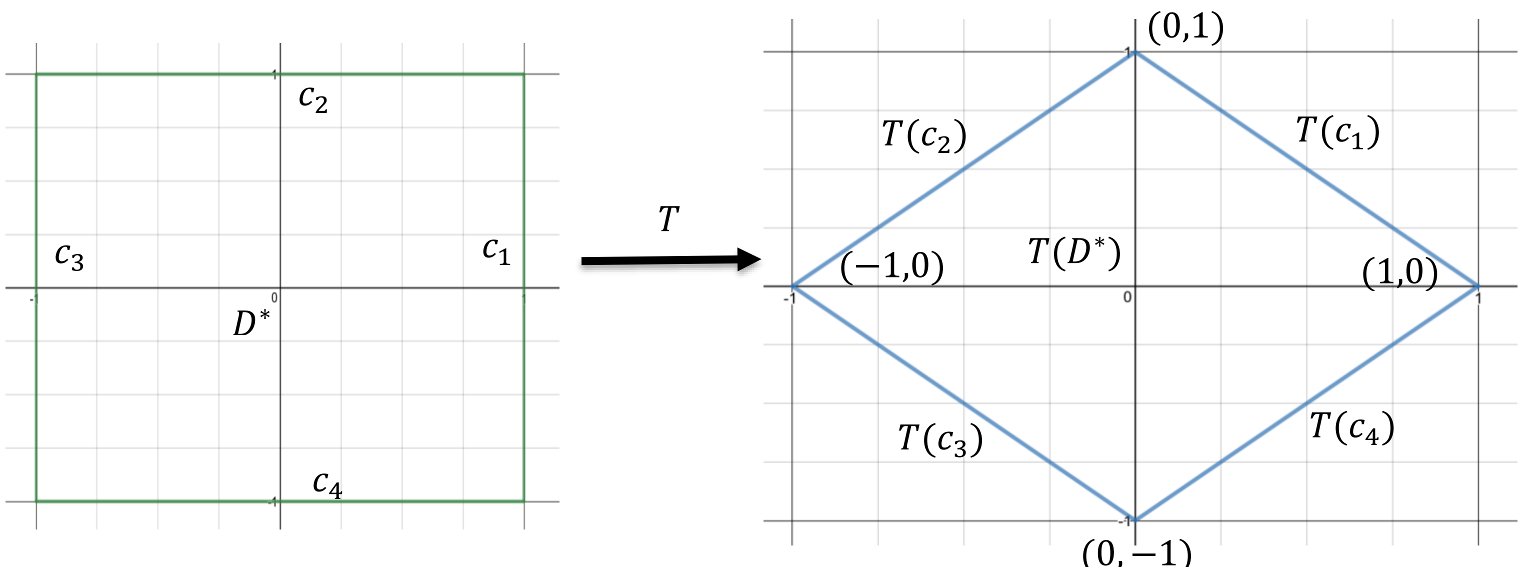
$c_3(t) = < -1, t > \qquad -1 \leq t \leq 1$

$c_4(t) = < t, -1 > \qquad -1 \leq t \leq 1.$

$T(c_2(t)) = \left(\frac{t-1}{2}, \frac{t+1}{2}\right)$; or $x - y = -1$,
a line segment starting at $(-1, 0)$ ending at $(0, 1)$.

$T(c_3(t)) = \left(\frac{-1-t}{2}, \frac{-1+t}{2}\right)$; or $x + y = -1$,
a line segment starting at $(0, -1)$ ending at $(-1, 0)$.

$T(c_4(t)) = \left(\frac{t+1}{2}, \frac{t-1}{2}\right)$; or $x - y = 1$,
a line segment starting at $(0, -1)$ ending at $(1, 0)$.



So T rotates D^* by $45°$ counterclockwise.

Def. A mapping $T: D^* \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is **one-to-one** if for:

$$(u, v), (u', v') \in D^*$$

$T(u, v) = T(u', v')$ implies that $u = u'$ and $v = v'$.

Thus, T is 1-1 if two different points in its domain are never mapped to the same point.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x^2, x^4 + y)$. Show that T is not 1-1.

T is not 1-1 because $T(1, 2) = T(-1, 2)$ (for example) but $(1, 2) \neq (-1, 2)$.

Ex. Consider the polar coordinate mapping:

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \text{ by } T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Show T is not 1-1 if the domain is all of $\mathbb{R}^2$.

Is T 1-1 if the domain is $D^* = [0, 1] \times [0, 2\pi)$?

$T(1, 0) = T(1, 2\pi)$ since:

$T(1, 0) = (1(\cos 0), 1(\sin 0)) = (1, 0)$

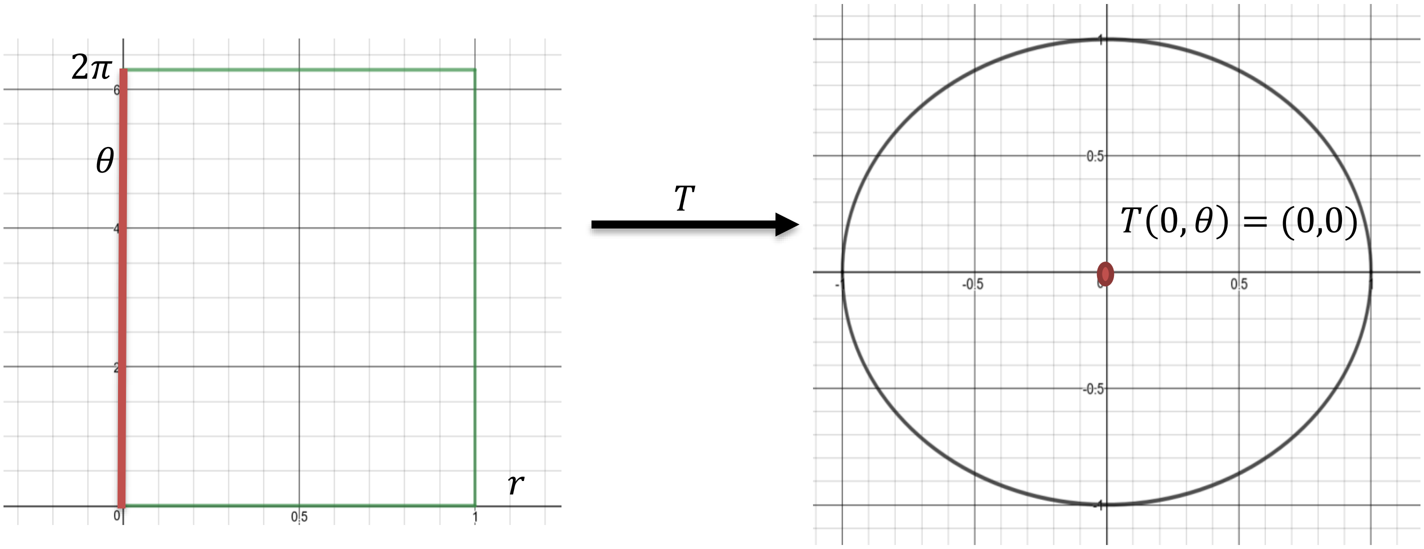
$T(1, 2\pi) = (1(\cos 2\pi), 1(\sin 2\pi)) = (1, 0)$

So T is not 1-1 on $\mathbb{R}^2$

If the domain is $D^* = [0,1] \times [0,2\pi)$ we still have:

$$T(0,\theta_1) = T(0,\theta_2) = (0,0) \text{ for any } 0 \le \theta_1, \theta_2 < 2\pi$$

So T is still not 1-1.



It **is** 1-1 if:

$$D^* = (0,1] \times [0,2\pi).$$

Ex. Show that $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = \left(\frac{x-y}{2}, \frac{x+y}{2}\right)$ is 1-1.

We must show that if $T(x,y) = T(x',y')$, then $x = x'$ and $y = y'$.

$$T(x,y) = \left(\frac{x-y}{2}, \frac{x+y}{2}\right)$$

$$T(x',y') = \left(\frac{x'-y'}{2}, \frac{x'+y'}{2}\right)$$

$$\left(\frac{x-y}{2}, \frac{x+y}{2}\right) = \left(\frac{x'-y'}{2}, \frac{x'+y'}{2}\right)$$

$$\frac{x-y}{2} = \frac{x'-y'}{2}$$

$$\frac{x+y}{2} = \frac{x'+y'}{2}$$

OR

$$\begin{array}{c} x - y = x' - y' \\ \underline{x + y = x' + y'} \\ 2x = 2x' \end{array}$$

$$x = x'$$

Subtracting the equations we get:

$$\begin{array}{c} x - y = x' - y' \\ \underline{x + y = x' + y'} \\ -2y = -2y' \end{array}$$

$$y = y'$$

Thus, $(x, y) = (x', y')$ and T is 1-1 on $\mathbb{R}^2$

Def. $T: D^* \subseteq \mathbb{R}^2 \to D$. The mapping T is **onto** D if for every point $(x, y) \in D$ there exists at least one point $(u, v) \in D^*$ such that $T(u, v) = (x, y)$.

Thus, T is onto if we can solve the equation: $T(u, v) = (x, y)$ where $(x, y) \in D$ and $(u, v) \in D^*$. If the solution is always unique, then T is also 1-1.

Ex. Determine if the following functions $T: \mathbb{R}^2 \to \mathbb{R}^2$ are 1-1 and/or onto.

a) $T(x, y) = (e^x, y)$

b) $T(r, \theta) = (r\cos\theta\,, r\sin\theta)$

c) $T(x, y) = (x^2, y)$

d) $T(x, y) = (\sqrt[3]{x}, \sqrt[3]{y})$

a) $e^x > 0$ so $T(x, y) = (e^x, y)$ can't be onto since, for example, there is no (x, y) such that $T(x, y) = (e^x, y) = (-1,1)$.

T is 1-1 on $\mathbb{R}^2$ since if $T(x, y) = T(x', y')$ and $(e^x, y) = (e^{x'}, y')$, then $e^x = e^{x'} \Rightarrow x = x'$, since $f(x) = e^x$ is strictly increasing, so it's 1-1. And $y = y'$ so we can say $(x, y) = (x', y')$.

b) T is onto since if $T(r, \theta) = (a, b)$ for any $(a, b) \in \mathbb{R}^2$, we have:

$a = r\cos\theta$ and $b = r\sin\theta$.

If $a = 0$, then $T\left(b, \frac{\pi}{2}\right) = (0, b)$.

If $a > 0$ then let $\theta = \tan^{-1}(\frac{b}{a})$, $r = \sqrt{a^2 + b^2}$, then

$$T(r, \theta) = (a, b).$$

If $a < 0$ then let $\theta = \pi + \tan^{-1}(\frac{b}{a})$, $r = \sqrt{a^2 + b^2}$, then

$$T(r, \theta) = (a, b).$$

T is not 1-1 since $T(0, \theta) = (0, 0)$ for all θ.

c) $T(x, y) = (x^2, y)$ is neither 1-1 nor onto.

$T(-1, y) = T(1, y)$ so T is not 1-1.

$x^2 \geq 0$, so there is no (x, y) such that $T(x, y) = (x^2, y) = (-1,1)$. Thus T is not onto.

d) $T(x,y) = \left(\sqrt[3]{x},\ \sqrt[3]{y}\right)$ is 1-1 and onto.

T is 1-1 since if $T(x,y) = T(x',y')$, then:

$$\left(\sqrt[3]{x}, \sqrt[3]{y}\right) = \left(\sqrt[3]{x'}, \sqrt[3]{y'}\right)$$

$$\sqrt[3]{x} = \sqrt[3]{x'} \qquad \sqrt[3]{y} = \sqrt[3]{y'}$$

$$x = x' \qquad y = y'$$

So $(x,y) = (x',y')$ and T is 1-1.

To show T is onto, let $(a,b) \in \mathbb{R}^2$. Then we must show that we can find $(x,y) \in \mathbb{R}^2$ such that $T(x,y) = (a,b)$.

$$T(x,y) = \left(\sqrt[3]{x}, \sqrt[3]{y}\right) = (a,b)$$

$$\sqrt[3]{x} = a \ ; \qquad \sqrt[3]{y} = b$$

$$x = a^3 \ ; \qquad y = b^3$$

So $T(a^3, b^3) = (a,b)$ and T is onto.

We can extend the notion of a change of variables to subsets of $D^* \subseteq \mathbb{R}^3$ as a map $T: D^* \to \mathbb{R}^3$.

To show $T: \mathbb{R}^3 \to \mathbb{R}^3$ is 1-1 you must show if:

$T(x,y,z) = T(x',y',z')$, then $x = x', y = y'$, and $z = z'$.

To show T is onto you must show given any $(a,b,c) \in \mathbb{R}^3$ that you can find x, y, z such that:

$$T(x,y,z) = (a,b,c).$$

Ex. Determine if the map $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x, y^3, xz)$ is 1-1 and/or onto.

T is not 1-1.

$$T(x,y,z) = T(x',y',z')$$
$$(x, y^3, xz) = (x', {y'}^3, x'z')$$

$x = x'$

$y^3 = y'^3 \implies y = y'$

$xz = x'z'$, but this does not imply that $z = z'$, since if $x = x' = 0$ the equation will be true for all values of z and z'.

In particular,

$$T(0,1,1) = (0,1,0) = T(0,1,3).$$

Thus T is not 1-1.

T is not onto.

Suppose that $T(x,y,z) = (a,b,c)$. Then we have:

$$(x, y^3, xz) = (a,b,c)$$

$x = a$

$y^3 = b \implies y = \sqrt[3]{b}$

$xz = c \implies az = c$

But if $a = 0$, and $c \neq 0$, then there is no z such that $az = c$.

In particular, there is no (x,y,z) such that $T(x,y,z) = (0,1,3)$.