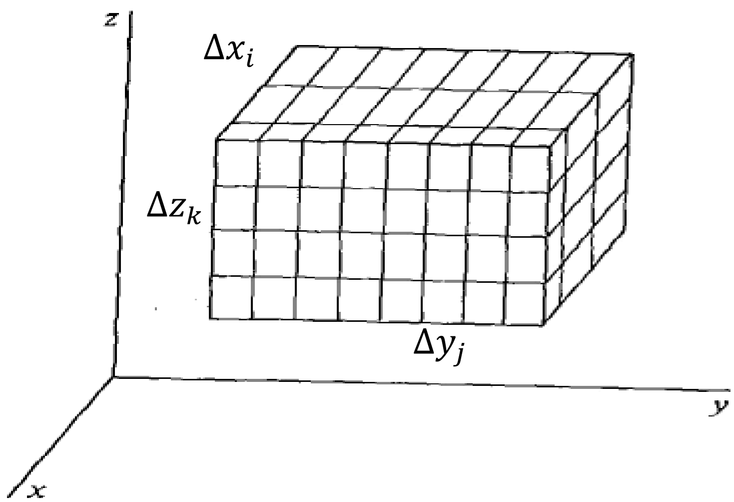
# Triple Integrals

We know how to integrate $y = f(x)$ over an interval and $z = f(x, y)$ over a region in the plane. Now we want to discuss integrating $w = f(x, y, z)$ over a rectangular solid box.

$$B = \{(x, y, z) |\ a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$

For $w = f(x, y, z)$ we can draw the region we're integrating over, but not the graph of $w = f(x, y, z)$ (we would need a 4 dimensional graph).




We divide $B$ into rectangular solids

And create a Riemann sum:

$$\sum f(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}; \quad \Delta V_{ijk} = \Delta x_i\ \Delta y_j\ \Delta z_k$$

$$\iiint_B f(x, y, z)\ dV = \lim_{l,m,n\to\infty} \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j, z_k)\ \Delta V_{ijk}\ .$$

Note: $\iiint_B 1\ dV =$Volume of the box B.

Fubini's Theorem

If $f$ is a continuous function on $B = [a, b] \times [c, d] \times [r, s]$, then:

$$\iiint_B f(x, y, z)\ dV = \int_{z=r}^{z=s} \left[ \int_{y=c}^{y=d} \left[ \int_{x=a}^{x=b} f(x, y, z)\ dx \right] dy \right] dz.$$

Ex. Evaluate $\iiint_B xy^2 e^z\ dV$ where $B$ is:

$$B = \{(x, y, z) |\ 0 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le \ln 2\}.$$

$$\iiint_B xy^2 e^z\ dV = \int_{z=0}^{z=\ln 2} \int_{y=0}^{y=3} \int_{x=0}^{x=2} xy^2 e^z\ dx\ dy\ dz$$

$$= \int_{z=0}^{z=\ln 2} \int_{y=0}^{y=3} \frac{x^2}{2} y^2 e^z \Big|_{x=0}^{x=2} dy\ dz$$

$$= \int_{z=0}^{z=\ln 2} \int_{y=0}^{y=3} 2y^2 e^z\ dy\ dz = \int_{z=0}^{z=\ln 2} \frac{2y^3}{3} e^z \Big|_{y=0}^{y=3} dz$$

$$= \int_{z=0}^{z=\ln 2} 18e^z\ dz = 18e^z|_{z=0}^{z=\ln 2} = 36 - 18 = 18.$$

We can calculate this integral in any order of $x, y,$ and $z$. For example:

$$\iiint_B xy^2 e^z\ dV = \int_{x=0}^{x=2} \int_{y=0}^{y=3} \int_{z=0}^{z=ln2} xy^2 e^z\ dz\ dy\ dx$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=3} xy^2 e^z|_{z=0}^{z=ln2}\ dy\ dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=3} xy^2(e^{ln2} - e^0)\ dy\, dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=3} xy^2(2 - 1)\ dy\, dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=3} xy^2\, dy\, dx$$

$$= \int_{x=0}^{x=2} \frac{xy^3}{3}\Big|_{y=0}^{y=3} dx$$

$$= \int_{x=0}^{x=2} 9x dx = \frac{9x^2}{2}\Big|_{x=0}^{x=2} = 18.$$

Ex. Evaluate $\iiint_B\ e^{(x+y+z)}\ dV$ where $B$ is:

$$B = \{(x, y, z)|\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 2\}.$$

$$\iiint_B\ e^{(x+y+z)}\ dV = \int_{z=0}^{z=2}\int_{y=0}^{y=1}\int_{x=0}^{x=1} e^{(x+y+z)}\, dx\, dy\, dz$$

$$= \int_{z=0}^{z=2}\int_{y=0}^{y=1} e^{(x+y+z)}\Big|_{x=0}^{x=1} dy\, dz$$

$$= \int_{z=0}^{z=2}\int_{y=0}^{y=1} [e^{(1+y+z)} - e^{(y+z)}] dy\, dz$$

$$= \int_{z=0}^{z=2} [e^{(1+y+z)} - e^{(y+z)}]\Big|_{y=0}^{y=1} dx$$

$$= \int_{z=0}^{z=2} \left[\left(e^{(2+z)} - e^{(1+z)}\right) - (e^{(1+z)} - e^z)\right] dz$$

$$= \int_{z=0}^{z=2} (e^{(2+z)} - 2e^{(1+z)} + e^z) dz$$

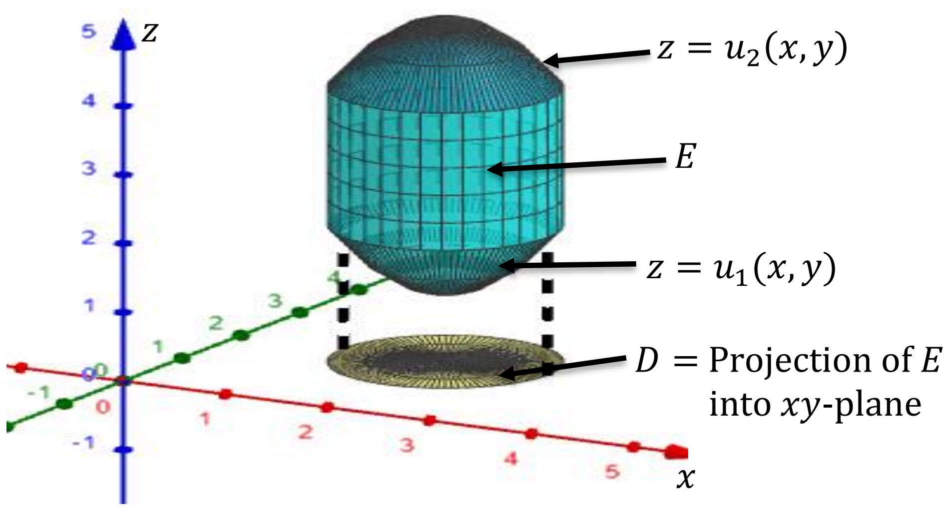
$$= \left(e^{(2+z)} - 2e^{(1+z)} + e^{z}\right)\Big|_{z=0}^{z=2}$$
$$= \left(e^4 - 2e^3 + e^2\right) - \left(e^2 - 2e + 1\right)$$
$$= e^4 - 2e^3 + 2e - 1.$$

Now we consider regions in $\mathbb{R}^3$ other than rectangular solids. Recall when we went from integrating over rectangles, $R$, in $\mathbb{R}^2$ to regions, $D$, bounded by curves:

$$\iint_R f(x,y)\,dA = \int_{y=c}^{y=d}\int_{x=a}^{x=b} f(x,y)\,dy\,dx$$

$$\rightarrow \quad \iint_D f(x,y)\,dA = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)dy\,dx.$$

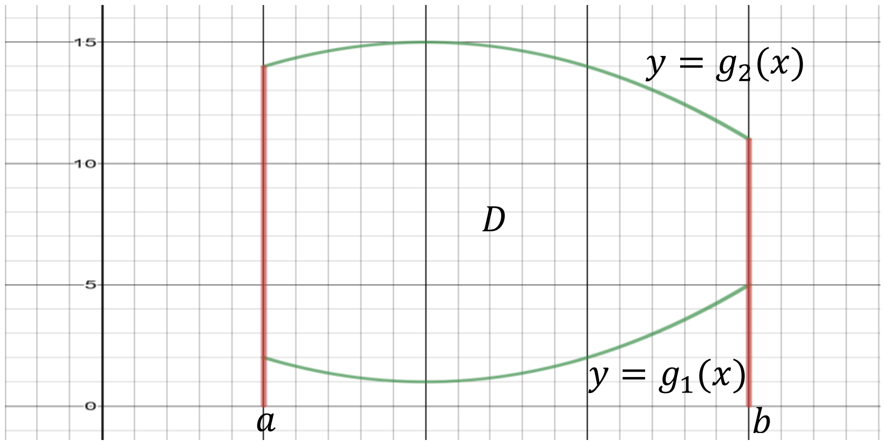
**Type 1**: Solid $E$ lies between 2 graphs of functions, $u_1(x,y)$ and $u_2(x,y)$.




$$\iiint_E f(x,y,z)dV = \iint_D \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\,dA$$

where: $E = \{(x,y,z) |\ (x,y) \in D\ , u_1(x,y) \le z \le u_2(x,y)\}$.

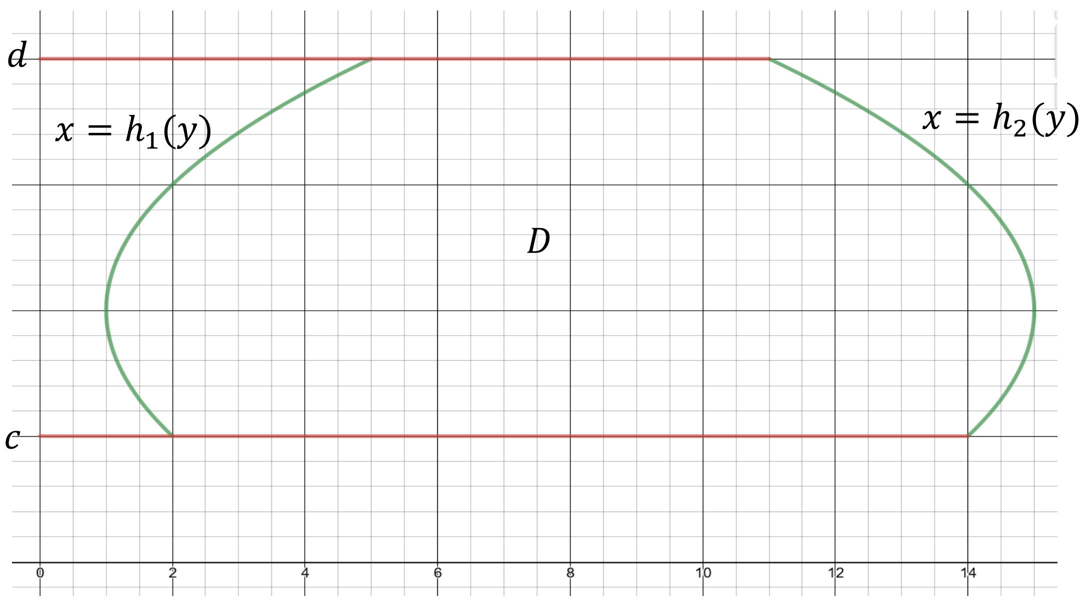
Now, if $D$ is bounded by $y = g_1(x)$ and $y = g_2(x)$,




then:

$$\iiint_E f(x,y,z)dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

If $D$ is bounded by $x = h_1(y)$ and $x = h_2(y)$,




then:

$$\iiint_E f(x,y,z)dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\, dz\, dx\, dy.$$

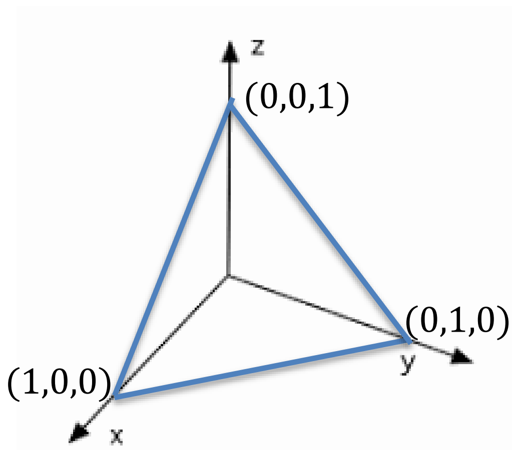
Ex. Evaluate $\iiint_E \ y\, dV$ where $E$ is the solid tetrahedron bounded by the 4 planes:

$$x = 0 \ , \ y = 0 \ , \ z = 0 \ , \ x + y + z = 1.$$

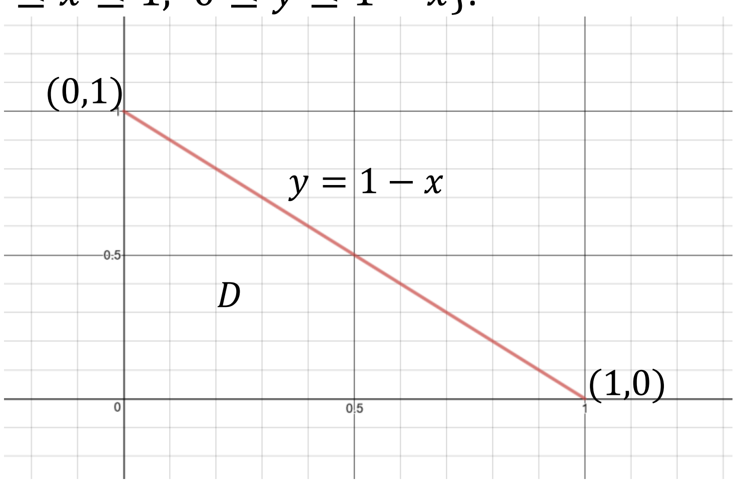
Draw two pictures:

1. $E$
2. The projection $D$ into the $xy$ plane

$x + y + z = 1$ intersects the $xy$ plane when $z = 0$, so $x + y = 1$. Thus, it intersects the $z$ axis when $x = y = 0 \ \Rightarrow z = 1$




$E = \{(x, y, z) | 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x, \ 0 \leq z \leq 1 - x - y\}$
$D = \{(x, y) | \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\}.$



$$\iiint_E y\,dV = \int_{x=0}^{x=1}\int_{y=0}^{y=-x+1}\int_{z=0}^{z=1-x-y} y\ dz\,dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=-x+1} yz\Big|_{z=0}^{z=1-x-y}\ dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=-x+1} y\,(1-x-y)\ dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=-x+1}(y-xy-y^2)\ dy\,dx$$

$$= \int_{x=0}^{x=1}\left(\frac{y^2}{2}-\frac{xy^2}{2}-\frac{y^3}{3}\right)\Big|_{y=0}^{y=-x+1}\ dx$$

$$= \int_{x=0}^{x=1}\left[\frac{(-x+1)^2}{2}-\frac{x(-x+1)^2}{2}-\frac{1}{3}(-x+1)^3\right]dx$$

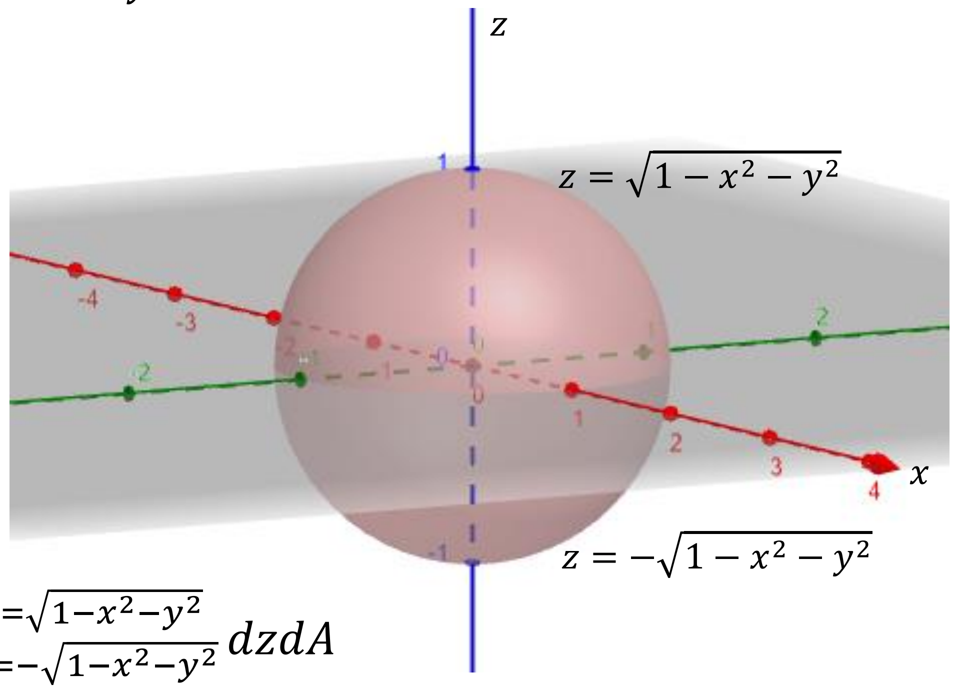
$$= \left[-\frac{(-x+1)^3}{6}+\frac{1}{12}(-x+1)^4\right]\Big|_0^1-\frac{1}{2}\int_{x=0}^{x=1}(x^3-2x^2+x)\ dx$$

$$= 0+0-\left[-\frac{1^3}{6}+\frac{1}{12}(1)^4\right]-\frac{1}{2}\left[\frac{x^4}{4}-\frac{2}{3}x^3+\frac{x^2}{2}\right]\Big|_0^1$$

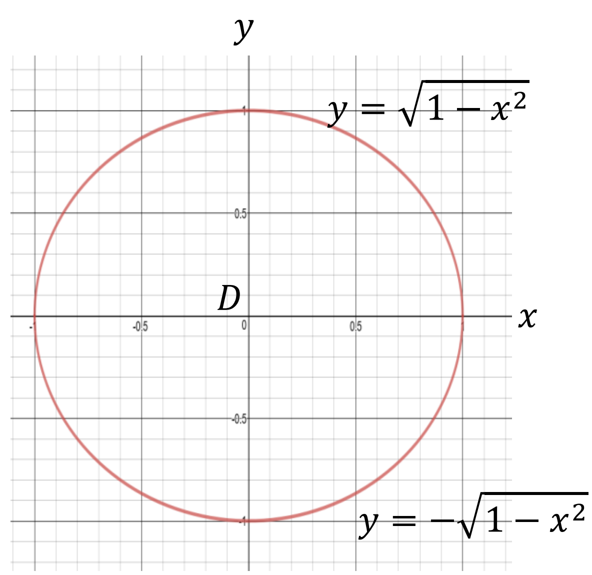
$$= -\left[-\frac{1}{6}+\frac{1}{12}\right]-\frac{1}{2}\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right) = -\left[-\frac{1}{12}\right]-\frac{1}{2}\left(\frac{3}{4}-\frac{2}{3}\right)$$

$$= \frac{1}{12}-\frac{1}{2}\left(\frac{9}{12}-\frac{8}{12}\right) = \frac{1}{12}-\frac{1}{24} = \frac{1}{24}\ .$$

Ex. Set up (but do not evaluate) the triple integral that represents the volume of the sphere $x^2 + y^2 + z^2 = 1$.




$$V = \iint_D \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} dz dA$$

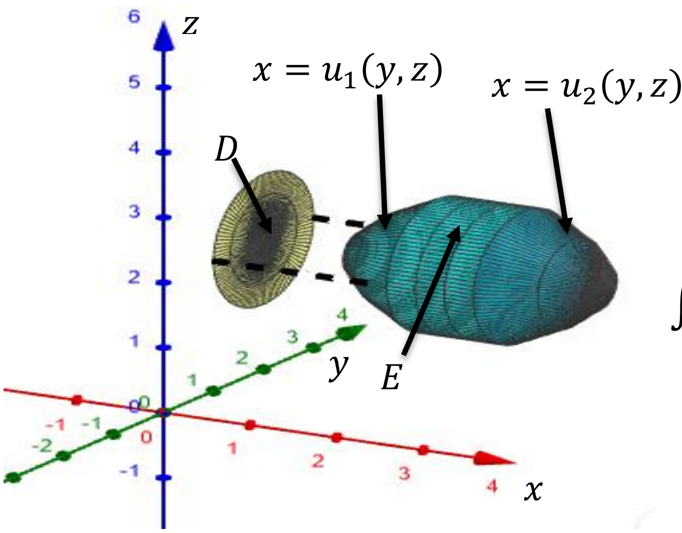



$$V = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} \, dz\, dy\, dx.$$

**Type 2:** A solid region of the form

$$E = \{(x, y, z) |\ (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$$

$D =$ projection of $E$ on to the $yz$ plane:

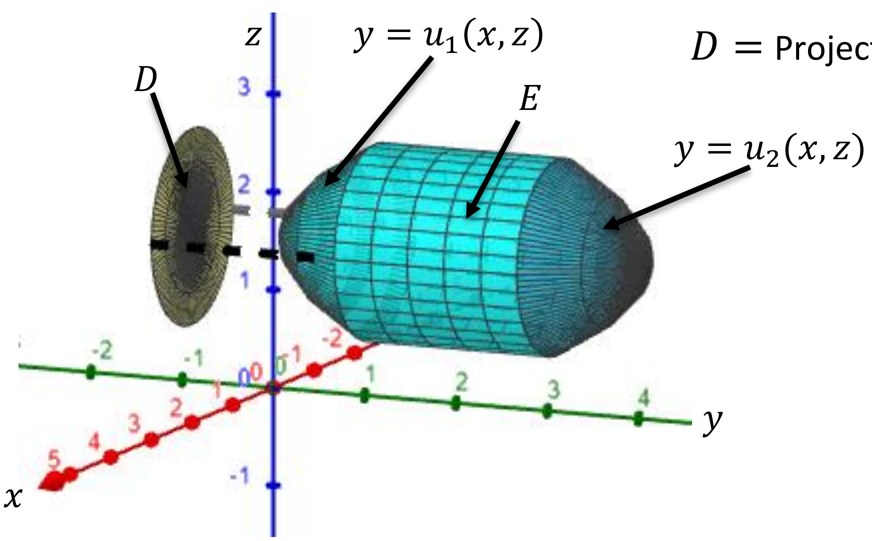



$$\iiint_E f(x, y, z)dV = \iint_D \int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x, y, z)\, dx\, dA$$

**Type 3:** Solid region of the form

$$E = \{(x, y, z) |\ (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

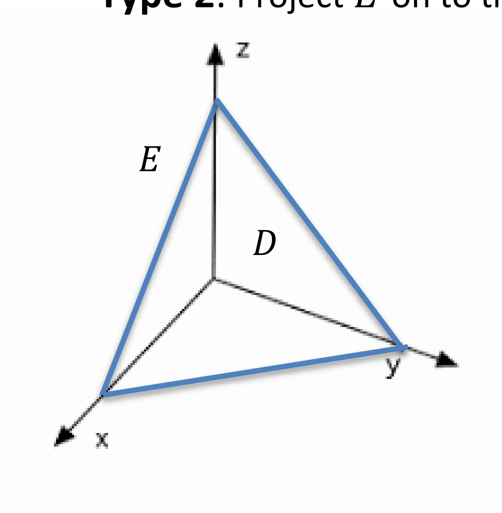
$D =$ Projection of $E$ on to the $xz$ plane:

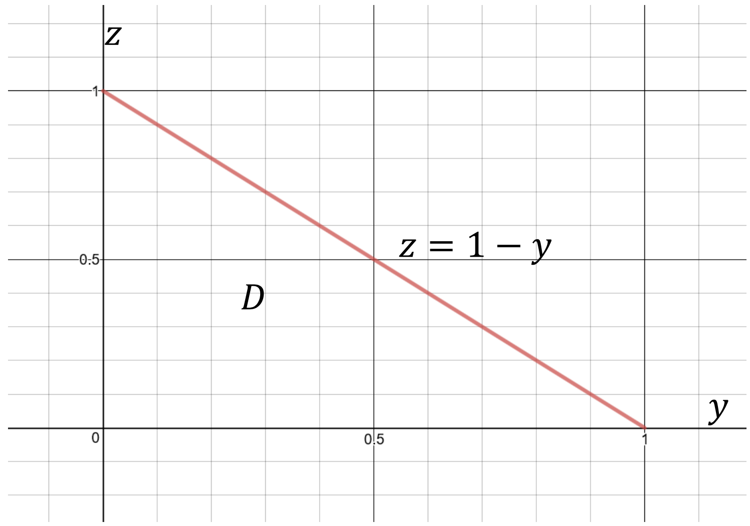



$$\iiint_E f(x, y, z)dV = \iint_D \int_{y=u_1(x,z)}^{y=u_2(x,z)} f(x, y, z)\, dy\, dA.$$

Ex. Find the limits of integration for $\iiint_E y\,dV$ where $E$ is the solid tetrahedron bounded by the planes: $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$ by treating it as a type 2 then a type 3 solid.

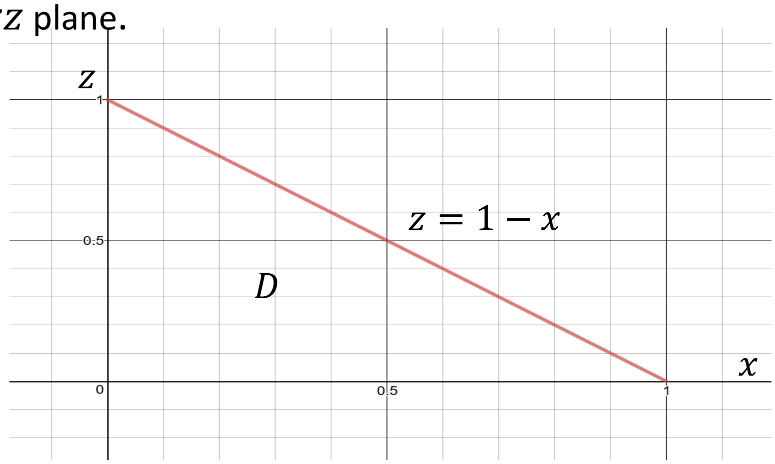
**Type 2**: Project $E$ on to the $yz$ plane.




$$\iiint_E y\,dV = \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \int_{x=0}^{x=1-y-z} y\,dx\,dz\,dy.$$

**Type 3**: Project $E$ onto the $xz$ plane.

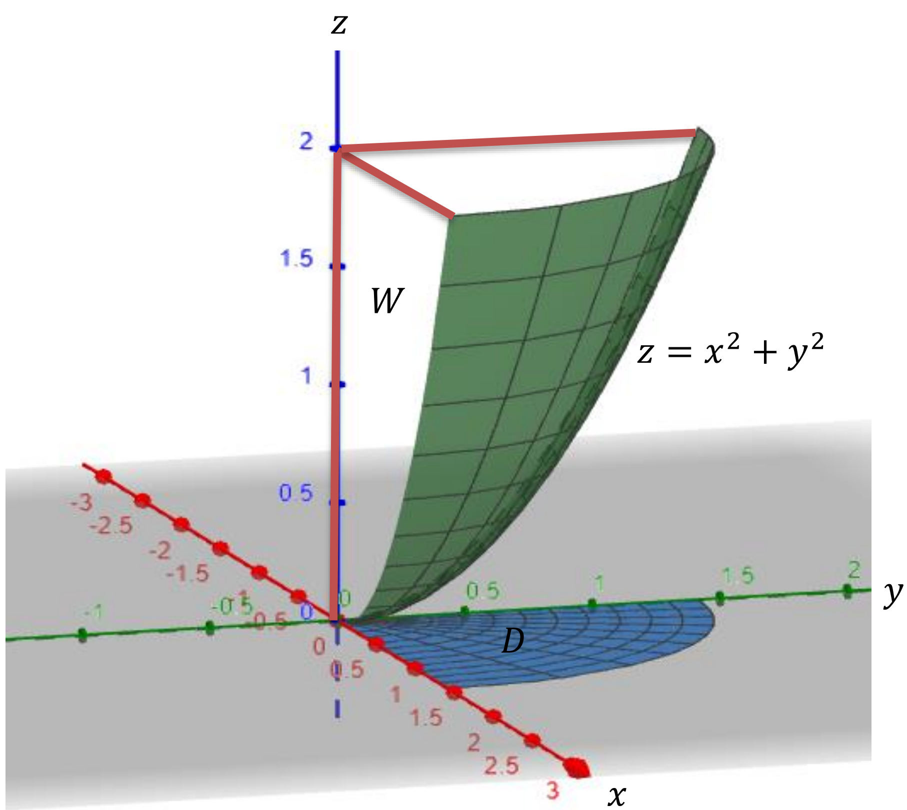



$$\iiint_E y\,dV = \int_{x=0}^{x=1} \int_{z=0}^{z=1-x} \int_{y=0}^{y=1-x-z} y\,dy\,dz\,dx$$

Ex. Let $W$ be the region bounded by the planes: $x = 0,\ y = 0$, and $z = 2$, and the surface $z = x^2 + y^2$ and lying in the first quadrant, i.e. $x \geq 0$, $y \geq 0$. Compute:

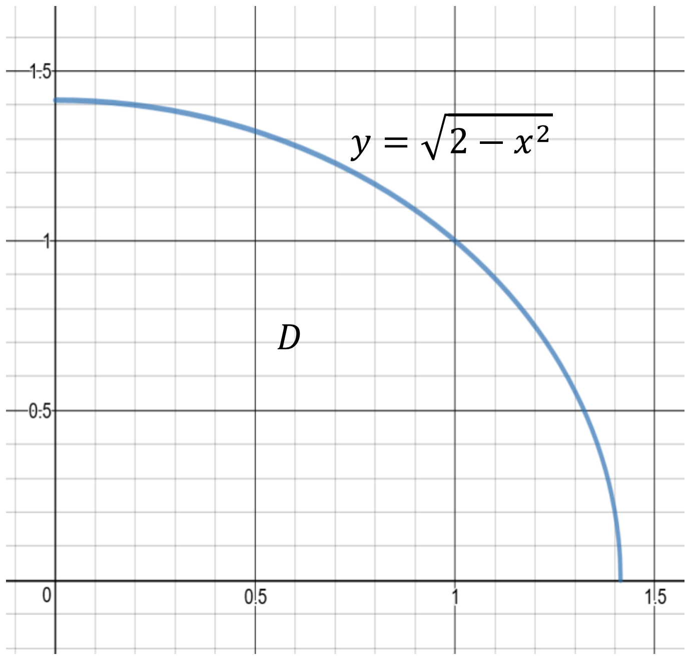
$$\iiint_W y\, dV.$$

Start by sketching the solid, $W$. $W$ is the region "inside" the bowl bounded on top by $z = 2$, below by $z = x^2 + y^2$, and on the sides by $x = 0$ (the $yz$ plane) and $y = 0$ (the $xz$ plane).



The projection of this solid into the $xy$ plane is the region in the first quadrant where: $x^2 + y^2 \leq 2$.

For this region, the "top" curve is $y = \sqrt{2 - x^2}$ and the "bottom" curve is $y = 0$.




$$\iiint_W y\,dV = \iiint_W y\,dz\,dy\,dx = \iint_D \int_{z=x^2+y^2}^{z=2} y\,dz\,dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} \int_{z=x^2+y^2}^{z=2} y\,dz\,dy\,dx$$

$$\iiint_W y\,dz\,dy\,dx = \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} yz\Big|_{z=x^2+y^2}^{z=2} dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} (2y - x^2y - y^3)\,dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} y^2 - \frac{x^2y^2}{2} - \frac{y^4}{4}\Bigg|_{y=0}^{y=\sqrt{2-x^2}} dx$$

$$= \int_{x=0}^{x=\sqrt{2}} (2-x^2) - \frac{x^2}{2}(2-x^2) - \frac{(2-x^2)^2}{4} \; dx$$

$$= \int_{x=0}^{x=\sqrt{2}} (2-x^2)[1 - \frac{x^2}{2} - \frac{1}{4}(2-x^2)] \; dx$$

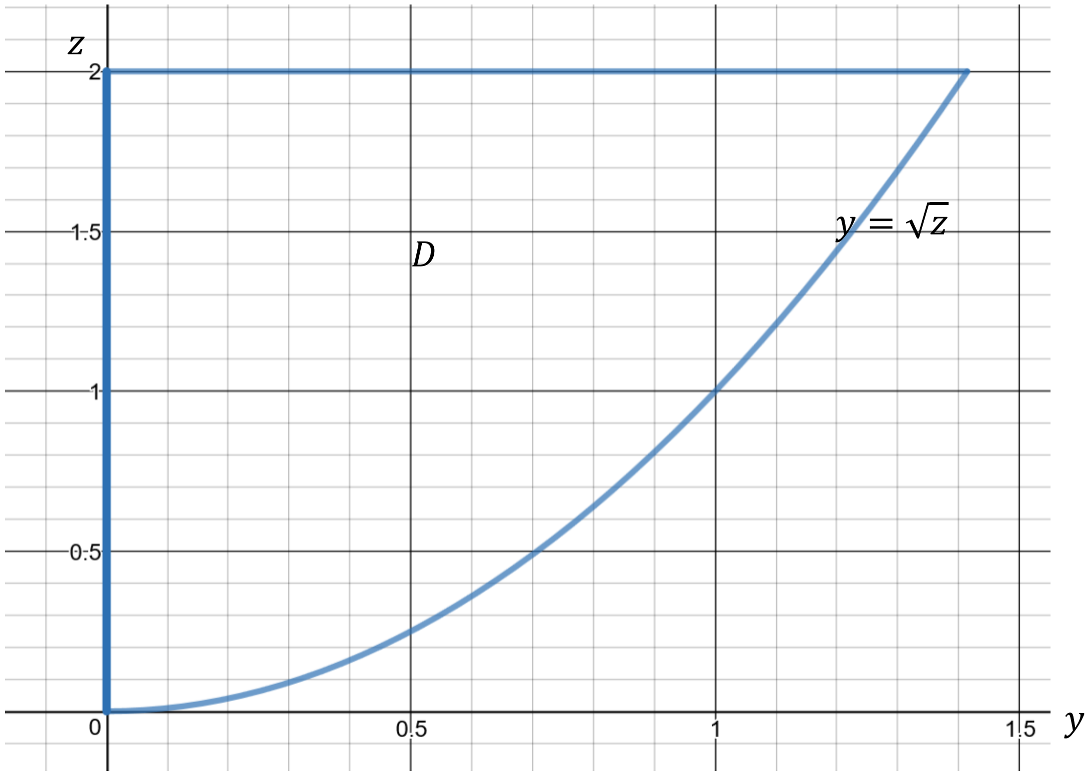
$$= \frac{1}{4}\int_{x=0}^{x=\sqrt{2}} 4 - 4x^2 + x^4 \; dx$$

$$= \tfrac{1}{4}(4x - \tfrac{4}{3}x^3 + \tfrac{1}{5}x^5\Big|_{x=0}^{x=\sqrt{2}} = \tfrac{8\sqrt{2}}{15}.$$

Here's a second way to do this problem:

Again, start by sketching the solid, $W$. However, this time we solve $z = x^2 + y^2$ for $x$ ( or $y$). Since $x \geq 0$ ; $x = \sqrt{z - y^2}$.

So $W$ is bounded by $x = \sqrt{z - y^2}$ on "top" and $x = 0$ on "bottom". To get the projection into the $yz$ plane, let $x = 0$ in $x = \sqrt{z - y^2}$ (i.e. $z = y^2$ or $y = \sqrt{z}$).




$$\iiint_W y\,dx\,dy\,dz = \iint_D \int_{x=0}^{x=(z-y^2)^{\frac{1}{2}}} y\,dx\,dy\,dz$$

$$= \int_{z=0}^{z=2} \int_{y=0}^{y=\sqrt{z}} \int_{x=0}^{x=(z-y^2)^{\frac{1}{2}}} y\,dx\,dy\,dz$$

$$= \int_{z=0}^{z=2} \int_{y=0}^{y=\sqrt{z}} xy\Big|_{x=0}^{x=(z-y^2)^{\frac{1}{2}}} dy\,dz$$

$$= \int_{z=0}^{z=2} \int_{y=0}^{y=\sqrt{z}} y(z-y^2)^{\frac{1}{2}}\, dy\, dz = \int_{z=0}^{z=2} -\frac{1}{3}(z-y^2)^{\frac{3}{2}}\Big|_{y=0}^{y=\sqrt{z}}\, dz$$

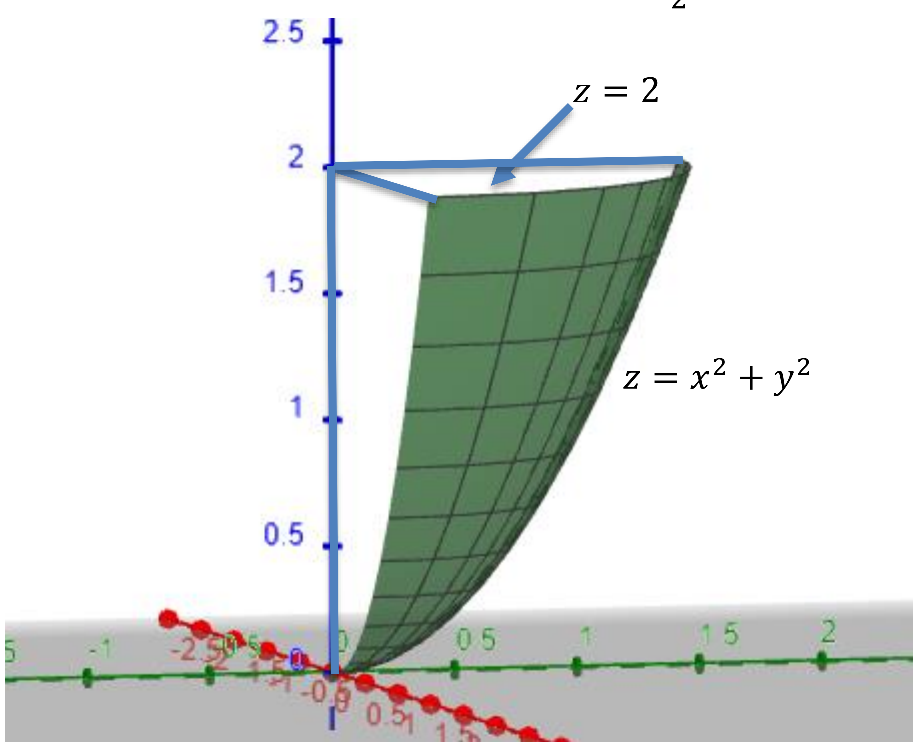
$$= \frac{1}{3} \int_{z=0}^{z=2} z^{\frac{3}{2}}\ dz$$

$$= \left(\frac{1}{3}\right)\left(\frac{2}{5}\right) z^{\frac{5}{2}}\Big|_{z=0}^{z=2} = \frac{2}{15}(2)^{\frac{5}{2}}$$

$$= \frac{8\sqrt{2}}{15}.$$

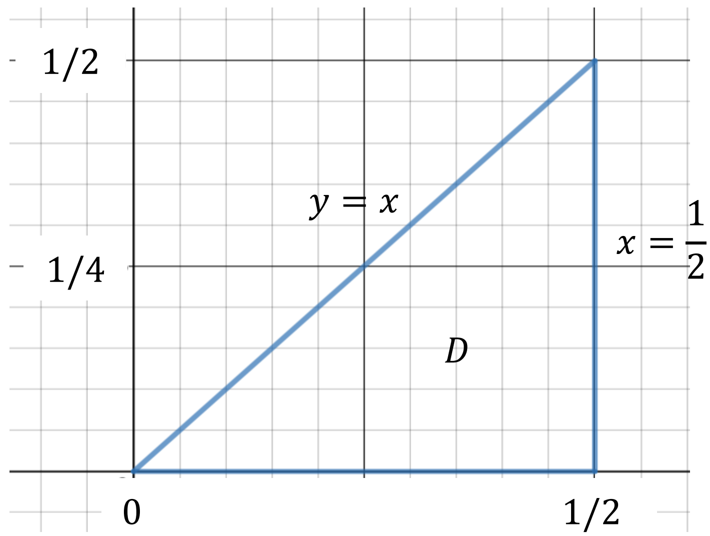
Ex. Find the volume of the solid bounded by the surface $z = x^2 + y^2$ and the planes: $z = 2, y = 0, x = \frac{1}{2}$, and $y = x$.

Start by drawing the solid, $W$. The "top" surface is $z = 2$ and the "bottom" surface is $z = x^2 + y^2$. $W$ is also cut by the planes $y = 0,\ x = \frac{1}{2}$, and $y = x$.



$$V = \iiint_W dV = \iint_D \int_{z=x^2+y^2}^{z=2} dz\, dy\, dx$$

The projection of $D$ into the $xy$ plane is bounded by $y = 0,\ x = \frac{1}{2}$, and $y = x$.




$$V = \int_{x=0}^{x=1/2} \int_{y=0}^{y=x} \int_{z=x^2+y^2}^{z=2} dz\, dy\, dx$$

$$= \int_{x=0}^{x=1/2} \int_{y=0}^{y=x} z\Big|_{z=x^2+y^2}^{z=2} dy\, dx = \int_{x=0}^{x=1/2} \int_{y=0}^{y=x} (2 - x^2 - y^2)dy\, dx$$

$$= \int_{x=0}^{x=1/2} (2y - x^2y - \frac{y^3}{3})\Big|_{y=0}^{y=x} dx = \int_{x=0}^{x=1/2} \left(2x - x^3 - \frac{x^3}{3}\right) dx$$

$$= \int_{x=0}^{x=1/2} (2x - \frac{4x^3}{3})\, dx = (x^2 - \frac{x^4}{3})\Big|_{x=0}^{x=\frac{1}{2}} = \frac{11}{48}.$$