Reversing the Order of Integration

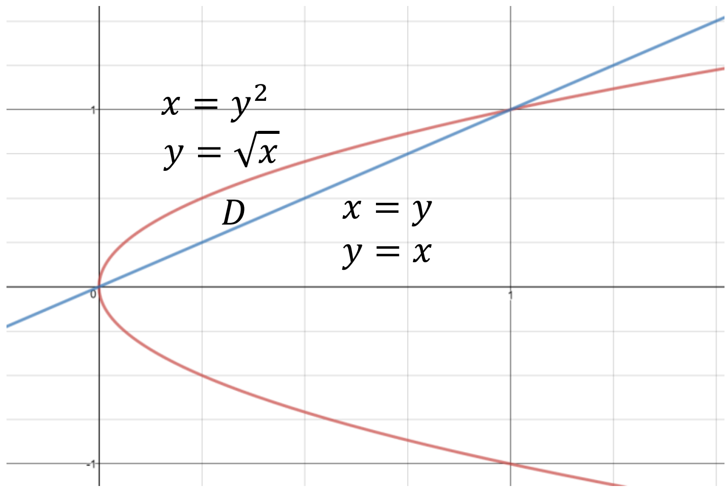
We could have also done an earlier example:

Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region D in the x, y plane bounded by $x = y^2$ and $y = x$.
by doing the following:

$$x = y^2 \quad \rightarrow \pm\sqrt{x} = y$$

Here, we are reversing the order of integration from our first approach.

$$x = \sqrt{x}$$
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0, \quad x = 1$$



$$\int_{x=0}^{x=1}\int_{y=x}^{y=\sqrt{x}} (3x^2 + y^2)\, dy\, dx = \int_{x=0}^{x=1} (3x^2y + \frac{y^3}{3})\Big]_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_{x=0}^{x=1} [\left(3x^2\sqrt{x} + \frac{x^{\frac{3}{2}}}{3}\right) - \left(3x^3 + \frac{x^3}{3}\right)]dx$$

$$= \int_{x=0}^{x=1} (3x^{\frac{5}{2}} + \frac{1}{3}x^{\frac{3}{2}} - \frac{10}{3}x^3)dx$$

$$= [3\left(\frac{2}{7}\right)x^{\frac{7}{2}} + \frac{1}{3}\left(\frac{2}{5}\right)x^{\frac{5}{2}} - \frac{5}{6}x^4]|_0^1$$

$$= \frac{6}{7} + \frac{2}{15} - \frac{5}{6} = \frac{11}{70}.$$

Ex. Find the volume of the tetrahedron bounded by the planes:

$$z = x, \qquad y = 2x, \qquad x + y = 3, \qquad z = 0.$$

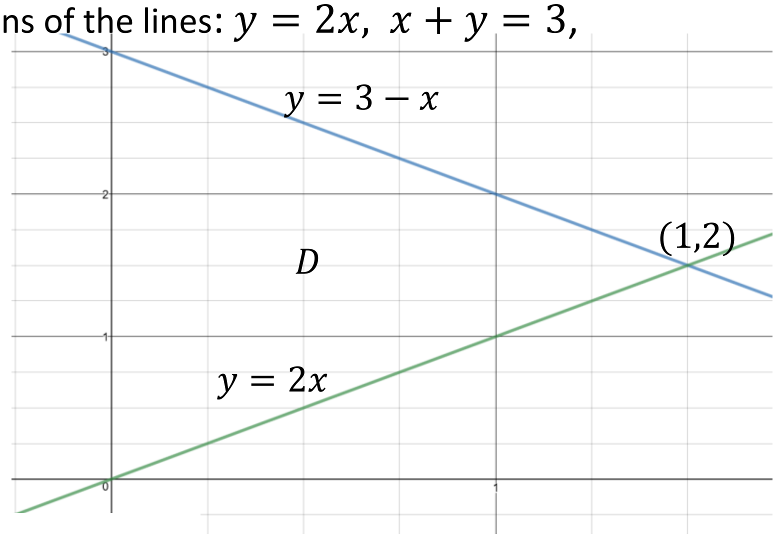
Draw the 3 dimensional solid and the region D, which lies below the solid in the xy plane. But first let's draw D:

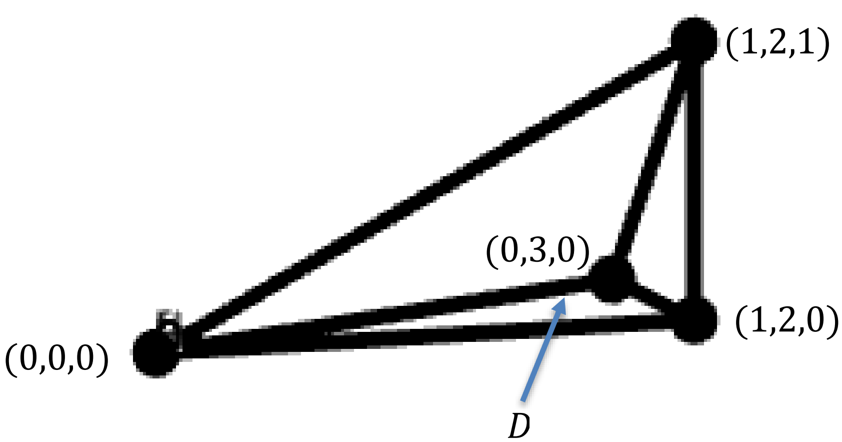
To find D, we need to find where each of the four planes intersects the xy plane.

$z = 0$ is the xy plane.
$y = 2x$ and $x + y = 3$ are the intersections with the xy plane
$z = x$ intersects the xy plane when $z = 0$:
i.e. when $x = 0$; y- axis

We need to find the intersections of the lines: $y = 2x,\ x + y = 3$, and $x = 0$.

$y = 2x$
$x + y = 3$
$x + 2x = 3$
$3x = 3$
$x = 1,\ \ y = 2.$





$$V = \iint_D z\, dA = \int_{x=0}^{x=1} \int_{y=2x}^{y=-x+3} x\, dy\, dx$$

$$= \int_0^1 xy\Big|_{y=2x}^{y=-x+3} dx = \int_0^1 [x(-x+3) - x(2x)]dx$$

$$= \int_0^1 (-3x^2 + 3x)dx$$

$$= -x^3 + \tfrac{3}{2}x^2\Big|_0^1 = -1 + \tfrac{3}{2} = \tfrac{1}{2}.$$

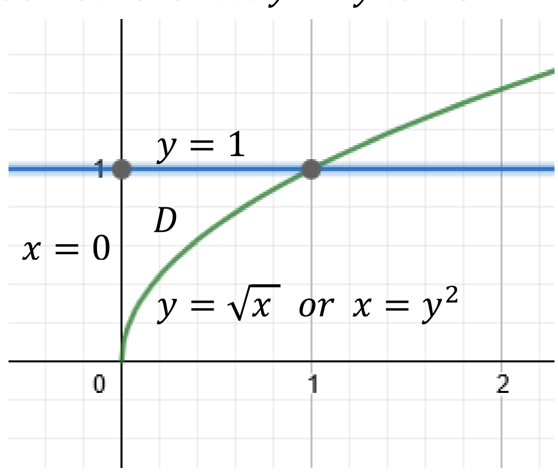
This could also be done as an x-simple integral by reversing the order of integration (but it's harder that way).

$$\iint_D x\, dA = \int_{y=0}^{y=2} \int_{x=0}^{x=\frac{1}{2}y} xdx\, dy + \int_{y=2}^{y=3} \int_{x=0}^{x=3-y} xdx\, dy$$

Sometimes it's difficult (or impossible) to evaluate a double integral when given an iterated integral but is easier to do if you change the order of integration.

Ex. Evaluate $\int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} (\cos y^3)\, dy\, dx.$

There is no elementary antiderivative for $\cos y^3$ in y terms.
Draw region of integration:



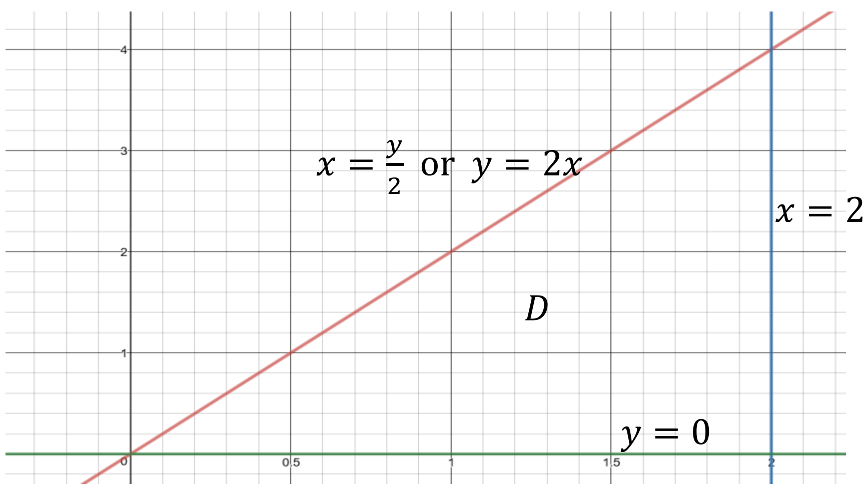
Now reverse the order of integration:

$$\int_{x=0}^{x=1}\int_{y=\sqrt{x}}^{y=1}(\cos y^3)\,dy\,dx = \int_{y=0}^{y=1}\int_{x=0}^{x=y^2}(\cos y^3)\,dx\,dy$$

$$= \int_{y=0}^{y=1} x\cos y^3\Big|_{x=0}^{x=y^2}\,dy = \int_{y=0}^{y=1}(y^2\cos y^3 - 0)\ dy$$

$$= \frac{1}{3}\sin y^3\Big|_{y=0}^{y=1} = \frac{1}{3}\sin(1) - \frac{1}{3}\sin(0) = \frac{1}{3}\sin(1).$$

Ex. Evaluate $\int_{y=0}^{y=4}\int_{x=y/2}^{x=2}(e^{x^2})\,dx\,dy$ by reversing the order of integration. Start by drawing the region over which we're integrating.



$$\int_{y=0}^{y=4}\int_{x=y/2}^{x=2}(e^{x^2})\,dx\,dy = \int_{x=0}^{x=2}\int_{y=0}^{y=2x}(e^{x^2})dy\,dx$$

$$= \int_{x=0}^{x=2} ye^{x^2}\Big|_{y=0}^{y=2x}\,dx$$

$$= \int_{x=0}^{x=2} 2xe^{x^2}\,dx$$

$$= e^{x^2}\Big|_{x=0}^{x=2}$$

$$= e^4 - e^0 = e^4 - 1.$$

Ex. Evaluate $\int_{y=0}^{y=1} \int_{x=0}^{x=\cos^{-1} y} (\sin x)\sqrt{1+\sin^2 x}\ dx\, dy$.

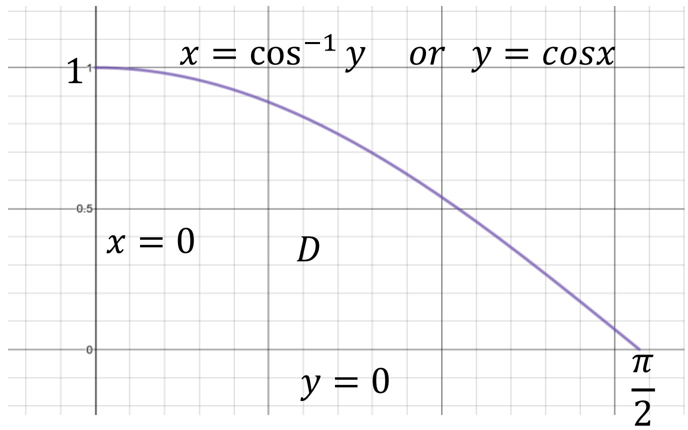
Although it is possible to find an anti-derivative for $(\sin x)\sqrt{1+\sin^2 x}$, it's much easier if we reverse the order of integration:

$x = 0;\qquad y = 0$

$x = \cos^{-1} y\,;\qquad y = 1$

$y = 0;\qquad x = 0$

$y = \cos x\,;\qquad x = \frac{\pi}{2}$



$$\int_{y=0}^{y=1} \int_{x=0}^{x=\cos^{-1} y} (\sin x)\sqrt{1+\sin^2 x}\ dx\, dy$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} \int_{y=0}^{y=\cos x} (\sin x)\sqrt{1+\sin^2 x}\ dy\, dx$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} y \sin x\ (1+\sin^2 x)^{\frac{1}{2}}\Big|_{y=0}^{y=\cos x} dx$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} (\cos x)(\sin x)(1+\sin^2 x)^{\frac{1}{2}}\ dx$$

Let $u = 1+\sin^2 x \qquad x = 0 \ \Rightarrow u = 1$

$du = 2(\sin x)\cos x\, dx \qquad x = \frac{\pi}{2} \ \Rightarrow u = 2$

$\frac{1}{2} du = (\sin x)\cos x\, dx$

$$= \frac{1}{2}\int_{u=1}^{u=2} u^{\frac{1}{2}}\ du = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) u^{\frac{3}{2}}\Big|_{u=1}^{u=2} = \frac{1}{3}\left(2\sqrt{2}-1\right).$$

Properties of Double Integrals:

1.

$$\iint_D f(x,y) + g(x,y)\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

2.

$$\iint_D c\, f(x,y)\, dA = c \iint_D f(x,y)\, dA$$

3. If $f(x,y) \geq g(x,y)$, then:

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$

4. If $D = D_1 \cup D_2$, where $D_1 \cap D_2 = \emptyset$:

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

5.

$$\iint_D 1\, dA = \text{area of } D$$

For 1 variable:

$$\int_a^b 1\, dx = b - a = \text{length of interval.}$$