The Dot Product, Length, and Distance

One way to multiply vectors is with the “dot product” (also called an inner product, or a scalar product).

Def. If $\vec{A} = < a_1, a_2, a_3 >$, $\vec{B} = < b_1, b_2, b_3 >$, then:

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = \boldsymbol{a_1 b_1 + a_2 b_2 + a_3 b_3},$$ in 2 dimensions this is $a_1 b_1 + a_2 b_2$.

Notice: So far, sums and differences of vectors resulted in a vector. The dot product of 2 vectors is a scalar, i.e. a number.

The definition for the dot product of two vectors works for n-dimensions as well. However, this multiplication only makes sense for 2 vectors: , $c \cdot \vec{A}$,where c is a constant, is meaningless.

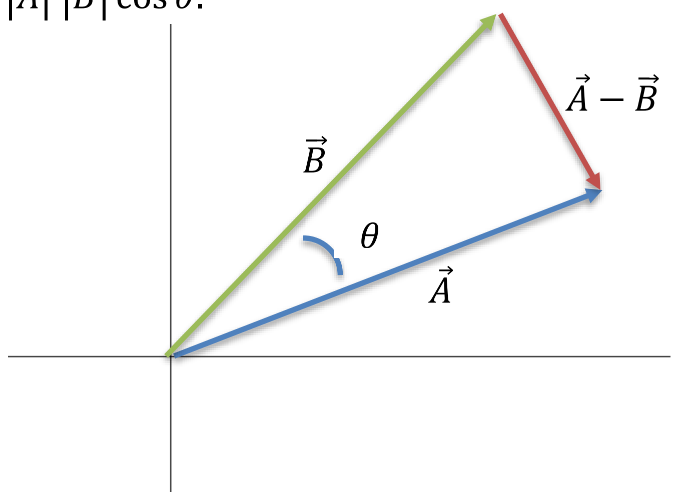
Properties of dot products: $\vec{A}$, $\vec{B}$, vectors, c a constant.

1. $\vec{A} \cdot \vec{A} = |\vec{A}|^2$
2. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
3. $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
4. $(c\vec{A}) \cdot \vec{B} = c(\vec{A} \cdot \vec{B}) = \vec{A} \cdot (c\vec{B})$
5. $\vec{0} \cdot \vec{A} = 0.$

Proof of 1:

$$\vec{A} \cdot \vec{A} = < a_1, a_2, a_3 > \cdot < a_1, a_2, a_3 > = a_1^2 + a_2^2 + a_3^2 = |\vec{A}|^2.$$

Theorem: $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$.



Law of cosines:

$$|\vec{A} - \vec{B}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}|\,|\vec{B}|\cos\theta$$

But we also know:

$$|\vec{A} - \vec{B}|^2 = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} - 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}$$

$$= |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

$$\Longrightarrow \qquad -2\vec{A} \cdot \vec{B} = -2|\vec{A}||\vec{B}|\cos\theta \;\Rightarrow\; \vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta.$$

Also,

$$\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \cos\theta.$$

Corollary: (The Cauchy-Schwarz Inequality) For any two vectors, $\vec{A}, \vec{B}$, we have $|\vec{A} \cdot \vec{B}| \le |\vec{A}||\vec{B}|$ and this equality holds if, and only if, $\vec{A}$ is a scalar multiple of $\vec{B}$ or either $\vec{A}$ or $\vec{B}$ is $\vec{0}$.

Ex. Suppose $|\vec{A}| = 3, |\vec{B}| = 5$, and the angle between $\vec{A}$ and $\vec{B}$ is $\frac{\pi}{4}$. Find $\vec{A} \cdot \vec{B}$.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$

$$\vec{A} \cdot \vec{B} = (3)(5)\cos\frac{\pi}{4} = 15\left(\frac{\sqrt{2}}{2}\right) = \frac{15\sqrt{2}}{2}.$$

Ex. Show that if $\vec{A} \perp \vec{B}$, then $\vec{A} \cdot \vec{B} = 0$.

$$\vec{A} \perp \vec{B} \Rightarrow \cos\theta = \cos\frac{\pi}{2} = 0$$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta = |\vec{A}||\vec{B}| 0 = 0.$$

The converse is also true: if $\vec{A}, \vec{B}$ are non-zero and $\vec{A} \cdot \vec{B} = 0 \Rightarrow \vec{A} \perp \vec{B}$.

Ex. Find the angle between $\vec{A} = < 2, -1, 3 >$ and $\vec{B} = < -4, 2, 1 >$.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta \Rightarrow \quad \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \cos\theta$$

$$\vec{A} \cdot \vec{B} = 2(-4) + (-1)(2) + (3)(1) = -8 - 2 + 3 = -7$$

$$|\vec{A}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

$$|\vec{B}| = \sqrt{(-4)^2 + 2^2 + 1^2} = \sqrt{21}$$

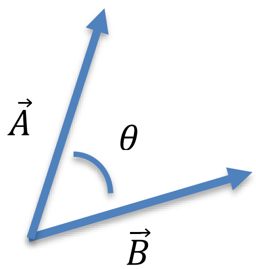
$$\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \frac{-7}{\sqrt{14(21)}} = \frac{-7}{7\sqrt{6}} = \frac{-1}{\sqrt{6}} = \cos\theta$$

$$\Rightarrow \theta \approx 1.99 \text{ radians} \approx 114°.$$

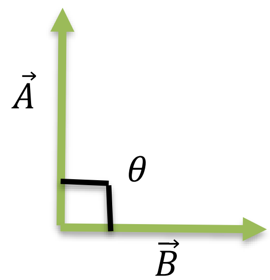
Notice that if

$$0 \le \theta < \frac{\pi}{2} \qquad \cos\theta > 0$$
$$\theta = \frac{\pi}{2} \qquad \cos\theta = 0$$
$$\frac{\pi}{2} < \theta \le \pi \qquad \cos\theta < 0$$

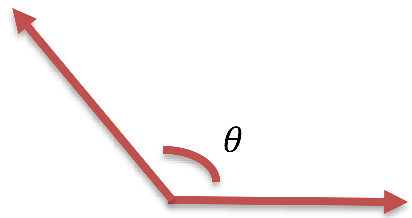
Thus we have:



$$0 \le \theta < \frac{\pi}{2}$$
$$\vec{A}\cdot\vec{B} > 0$$



$$\theta = \frac{\pi}{2}$$
$$\vec{A}\cdot\vec{B} = 0$$



$$\frac{\pi}{2} < \theta \le \pi$$
$$\vec{A}\cdot\vec{B} < 0$$

Parallel Vectors:

If $\vec{A}$ and $\vec{B}$ point in the same direction:

$$\theta = 0\,; \quad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos 0 = |\vec{A}||\vec{B}|$$

If $\vec{A}$ and $\vec{B}$ point in the opposite direction:

$$\theta = \pi\,; \quad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\pi = -|\vec{A}||\vec{B}|.$$

So if $\vec{A} \parallel \vec{B}$ then $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|$ or $-|\vec{A}||\vec{B}|$.

Ex. Determine if the following vectors $\vec{A} = 2\vec{\imath} + 6\vec{\jmath} - 4\vec{k}$, $\vec{B} = -3\vec{\imath} - 9\vec{\jmath} + 6\vec{k}$ are parallel, orthogonal (i.e., perpendicular), or neither.

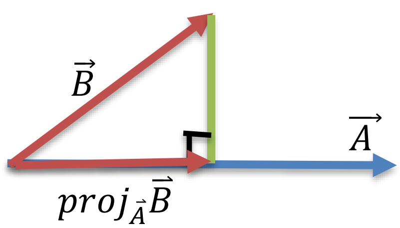
$$\vec{A}\cdot\vec{B} = -6 - 54 - 24 = -84$$

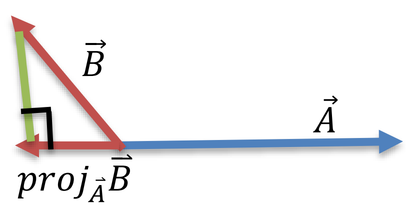
$$|\vec{A}| = \sqrt{2^2 + 6^2 + (-4)^2} = \sqrt{4 + 36 + 16} = \sqrt{56}$$

$$|\vec{B}| = \sqrt{(-3)^2 + (-9)^2 + 6^2} = \sqrt{9 + 81 + 36} = \sqrt{126}$$

$$|\vec{A}||\vec{B}| = \sqrt{56}\sqrt{126} = 84 = -\vec{A}\cdot\vec{B},$$ so $\vec{A}$ and $\vec{B}$ are parallel.

Def. $\overrightarrow{PS} = proj_{\vec{A}}\vec{B} =$ **vector projection of $\vec{B}$ onto $\vec{A}$. The scalar projection of $\vec{B}$ onto $\vec{A}$** (also called the component of $\vec{B}$ along $\vec{A}$) is defined to be the signed length of the vector projection of $\vec{B}$ onto $\vec{A}$.





Scalar projection of $\vec{B}$ onto $\vec{A} = \pm|proj_{\vec{A}}\vec{B}|$, notice that:

$$\cos\theta = \frac{\pm|proj_{\vec{A}}\vec{B}|}{|\vec{B}|}$$

Scalar Projection of $\vec{B}$ onto $\vec{A} = comp_{\vec{A}}\vec{B} = |\vec{B}|\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|}$
Vector Projection of $\vec{B}$ onto $\vec{A} = proj_{\vec{A}}\vec{B} = \left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|}\right)\frac{\vec{A}}{|\vec{A}|} = (\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2})\vec{A}$

Ex. Find the scalar projection of $\vec{B}$ onto $\vec{A}$ and the vector projection of $\vec{B}$ onto $\vec{A}$ if $\vec{A} = < -2, 1, 3 >, \vec{B} = < -2, -1, 4 >$.

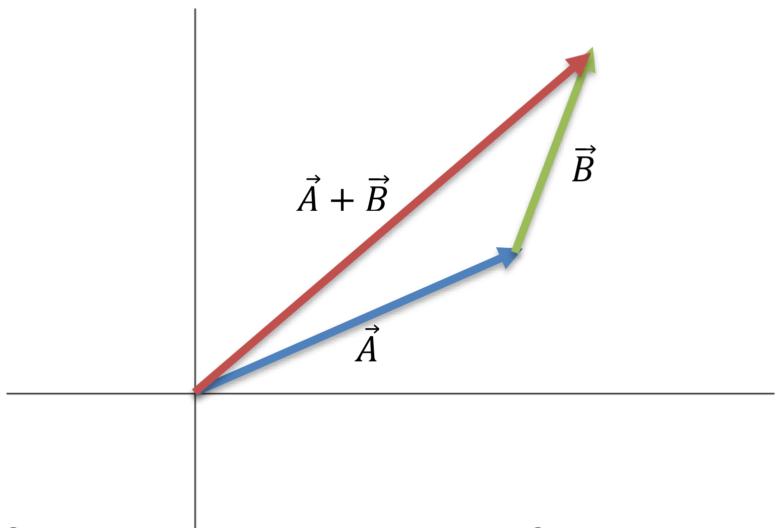
$$comp_{\vec{A}}\vec{B} = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|} = \frac{4-1+12}{\sqrt{(-2)^2+1^2+3^3}} = \frac{15}{\sqrt{14}}$$

$$proj_{\vec{A}}\vec{B} = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|^2}\vec{A} = \frac{15}{14} < -2, 1, 3 > = < -\frac{15}{7}, \frac{15}{14}, \frac{45}{14} > .$$

Theorem (Triangle Inequality): For vectors $\vec{A}$ and $\vec{B}$

$$\|\vec{A} + \vec{B}\| \leq \|\vec{A}\| + \|\vec{B}\|$$

Proof:



$$\|\vec{A} + \vec{B}\|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \|\vec{A}\|^2 + 2\vec{A} \cdot \vec{B} + \|\vec{B}\|^2$$

$$\leq \|\vec{A}\|^2 + 2\|\vec{A}\|\|\vec{B}\| + \|\vec{B}\|^2 \quad \text{(by the Cauchy-Schwarz Inequality)}$$

$$= (\|\vec{A}\| + \|\vec{B}\|)^2$$

So we can now write:

$$\|\vec{A} + \vec{B}\| \leq \|\vec{A}\| + \|\vec{B}\|.$$