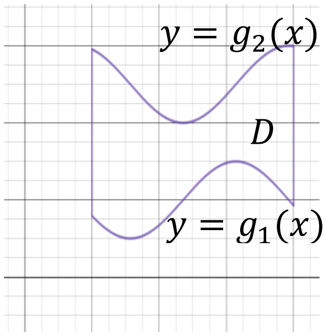
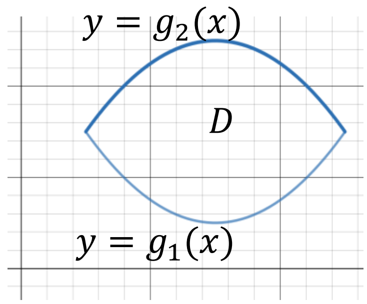
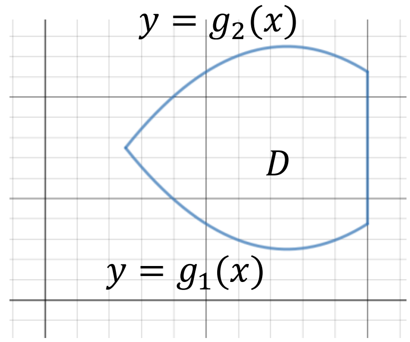
# The Double Integral over more General Regions

Instead of integrating over a rectangle, suppose the region looks like:

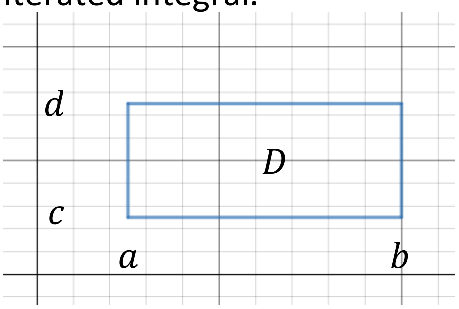


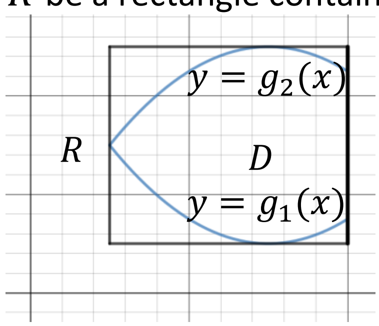

These regions are called ***y*-simple** (i.e. $g_1(x) \leq y \leq g_2(x)$).

How do we calculate $\iint_D f(x,y)\, dA$? When $D$ was a rectangle we did it with an iterated integral.




$\iint_D f(x,y)\, dA = \int_{x=a}^{x=b} [\int_{y=c}^{y=d} f(x,y)\, dy]\, dx$

For a region, $D$, bounded by $y = g_2(x),\ y = g_1(x),\ x = a,\ x = b,$
let $R$ be a rectangle containing $D$.




Define:

$F(x,y) = f(x,y)$ if $(x,y) \in D$
$= 0$ if $(x,y) \in R$ but not in $D$

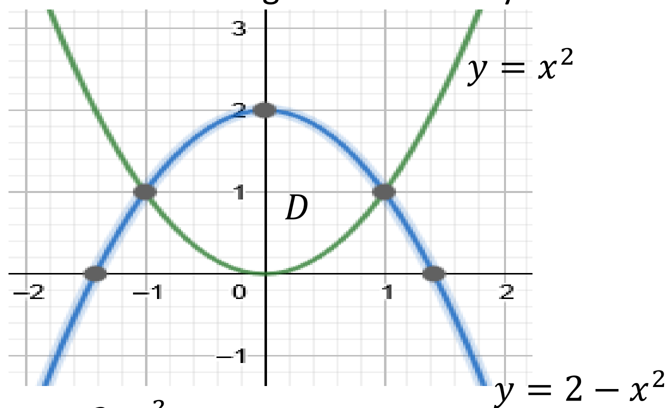
Define: $\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$. By Fubini's Theorem:

$$\iint_R F(x,y)\, dA = \int_a^b \int_c^d F(x,y)\, dy\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

Notice that $\int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy\, dx =$area between $y = g_1(x),\ y = g_2(x),$ where $a \le x \le b.$

Ex. Evaluate $\iint_D (x - 2y)\, dA$, where $D$ is the region bounded by:

$y = 2 - x^2,\ \ y = x^2.$




$2 - x^2 = x^2$

$2 = 2x^2$

$x = \pm 1$

$$\iint_D (x - 2y)\, dA = \int_{x=-1}^{x=1} \int_{y=x^2}^{y=2-x^2} (x - 2y)\, dy\, dx$$

$$= \int_{-1}^{1} (xy - y^2)\Big|_{y=x^2}^{y=2-x^2} dx$$

$$= \int_{-1}^{1} (x(2 - x^2) - (2 - x^2)^2) - (x(x^2) - (x^2)^2)\ dx$$

$$= \int_{-1}^{1} 2x - x^3 - (4 - 4x^2 + x^4) - (x^3 - x^4)\ dx$$

$$= \int_{-1}^{1} 2x - x^3 - 4 + 4x^2 - x^4 - x^3 + x^4\ \ dx$$

$$= \int_{-1}^{1} -2x^3 + 4x^2 + 2x - 4\ \ dx$$

$$= -\tfrac{1}{2}x^4 + \tfrac{4}{3}x^3 + x^2 - 4x\Big]_{-1}^{1}$$

$$= \left(-\tfrac{1}{2}(1) + \tfrac{4}{3} + 1 - 4\right) - \left(-\tfrac{1}{2}(1) - \tfrac{4}{3} + 1 + 4\right)$$

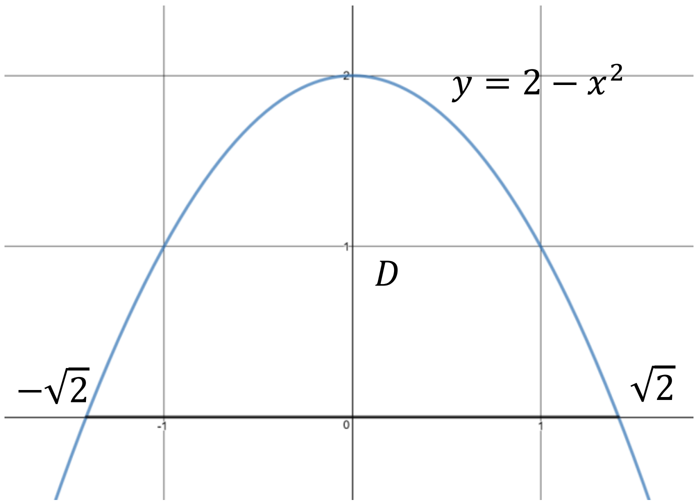
$$= -\tfrac{7}{2} + \tfrac{4}{3} - \left(\tfrac{9}{2} - \tfrac{4}{3}\right) = -8 + \tfrac{8}{3} = -\tfrac{16}{3}.$$

Ex. Find the limits of integration in the previous example if D Is bounded by

a. $y = 2 - x^2$ and the $x$ axis

b. $y = x$ and $y = x^2$.

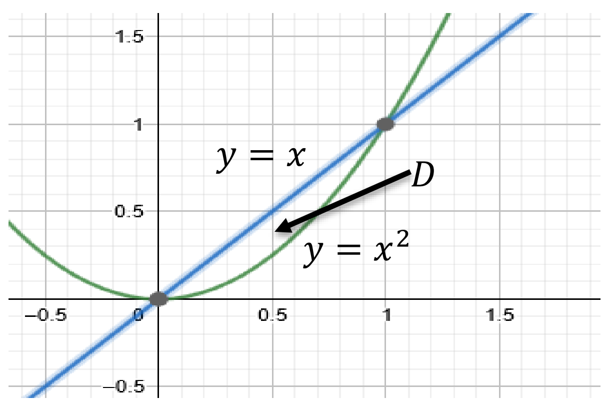
a. $y = 2 - x^2$ intersects the $x$ axis when $2 - x^2 = 0$ or $x = \pm\sqrt{2}$.




$$\iint_D (x - 2y)\, dA = \int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} (x - 2y)\, dy\, dx.$$

b. $y = x$ and $y = x^2$ intersect when

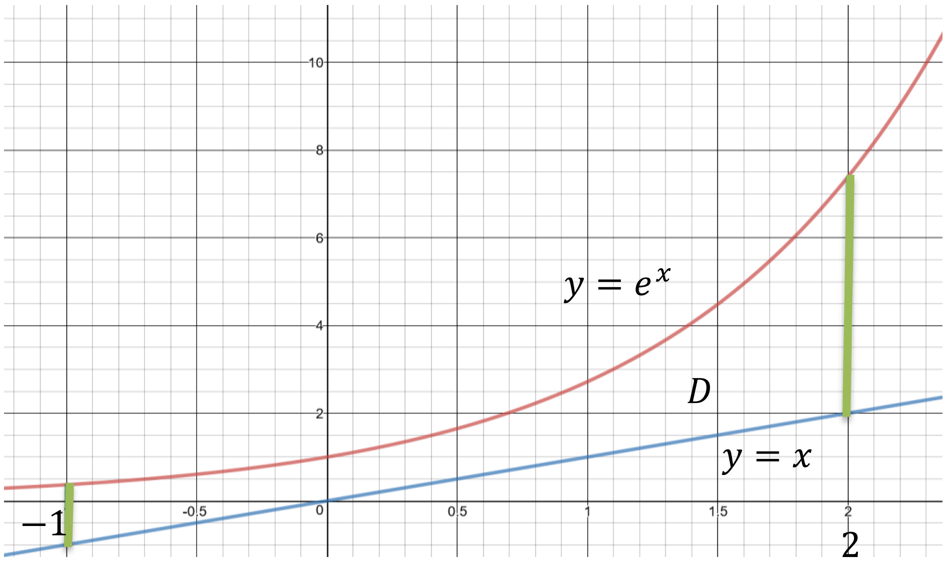
$$x = x^2$$
$$x - x^2 = 0$$
$$x(1 - x) = 0$$
$$x = 0,1$$




$$\iint_D (x - 2y)\, dA = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (x - 2y)\, dy\, dx.$$

Ex. Sketch the region $D$, over which the following integral is being taken:

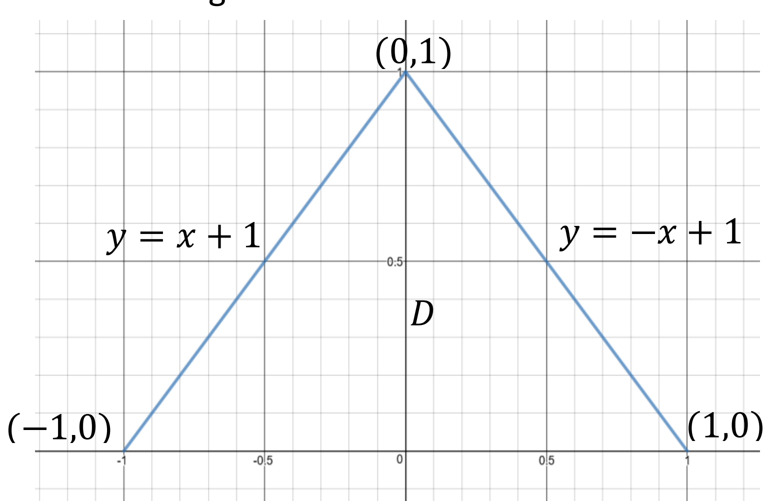
$$\int_{x=-1}^{x=2} \int_{y=x}^{y=e^x} (x+y)\, dy\, dx$$




Ex. Evaluate the following integral if the region $D$ is the interior of the triangle whose vertices are $(-1,0)$, $(1,0)$, and $(0,1)$.

$$\iint_D (x+2y)\, dA$$

1. Start by drawing $D$.
2. Find the equations of the sides of the triangle.
3. Notice that the "top curve" changes as $x$ goes from $-1$ to $1$, so we need to break this into two integrals.



Equations of the sides: $y = x + 1,\ y = -x + 1,\ y = 0.$

$$\iint_D (x + 2y)dA =$$

$$= \int_{x=-1}^{x=0} \int_{y=0}^{y=x+1} (x + 2y)\, dy\, dx + \int_{x=0}^{x=1} \int_{y=0}^{y=-x+1} (x + 2y)\, dy\, dx$$

$$= \int_{x=-1}^{x=0} (xy + y^2)|_{y=0}^{y=x+1}\, dx + \int_{x=0}^{x=1} (xy + y^2)|_{y=0}^{y=-x+1}\, dx$$

$$= \int_{x=-1}^{x=0} (x(x + 1) + (x + 1)^2)\, dx + \int_{x=0}^{x=1} [x(-x + 1) + (-x + 1)^2]\, dx$$

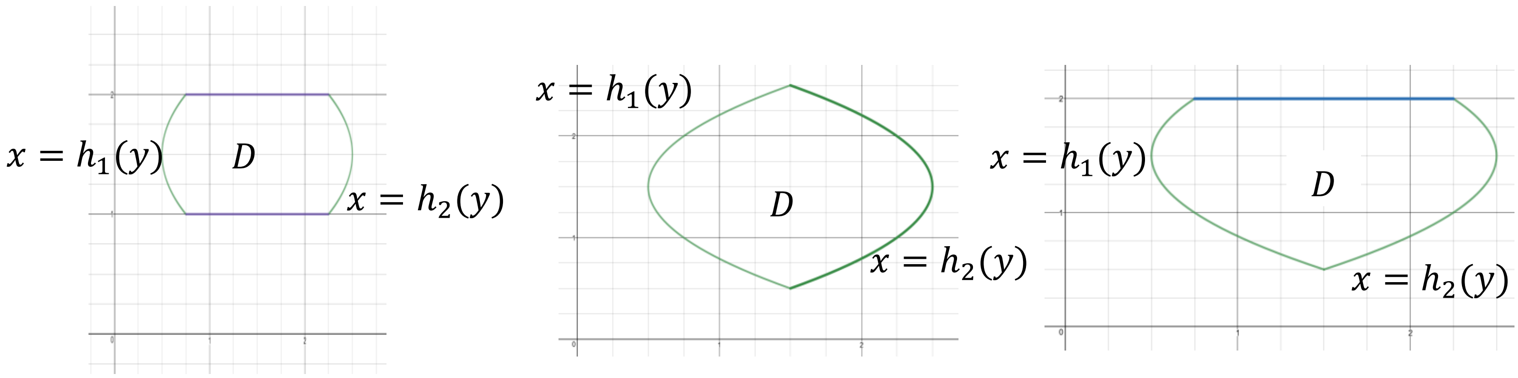
$$= \int_{x=-1}^{x=0} (2x^2 + 3x + 1)\, dx + \int_{x=0}^{x=1} (-x + 1)\, dx$$

$$= (\frac{2x^3}{3} + \frac{3x^2}{2} + x)\Big|_{x=-1}^{x=0} + \left(-\frac{x^2}{2} + x\right)\Big|_{x=0}^{x=1}$$

$$= 0 - \left(-\frac{2}{3} + \frac{3}{2} - 1\right) + \left(-\frac{1}{2} + 1\right) - 0$$

$$= \frac{1}{6} + \frac{1}{2} = \frac{2}{3}.$$

If $D$ looks like:



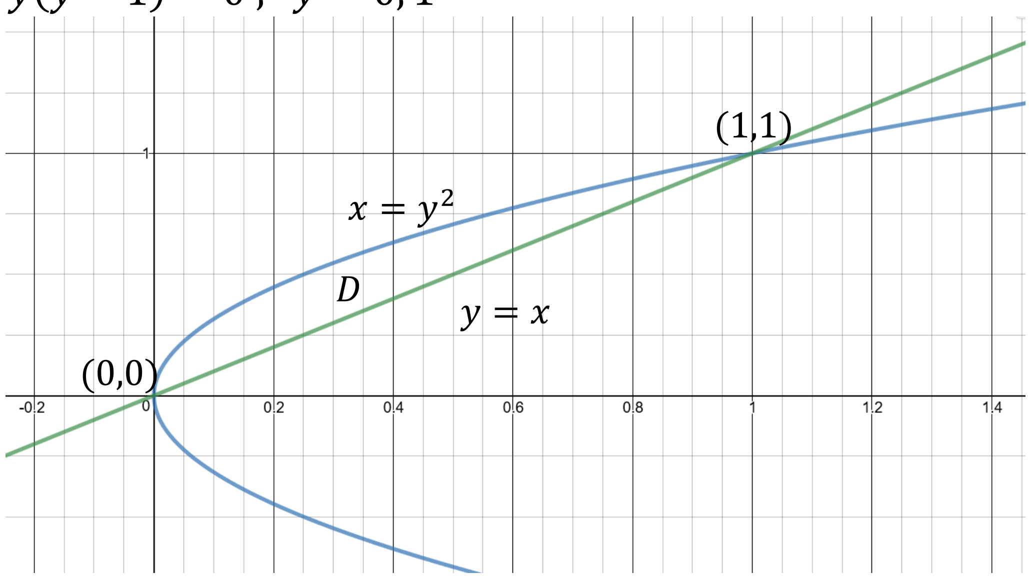

we call these regions ***x*-simple**, i.e. $h_1(y) \le x \le h_2(y)$ and

$$\iint_D f(x,y)\,dA = \int_{y=c}^{y=d} [\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,]dy.$$

Ex. Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region $D$ in the $x, y$ plane bounded by $x = y^2$ and $y = x$.

$y^2 = y \;\; \Rightarrow \;\; y^2 - y = 0$

$y(y-1) = 0\,; \;\; y = 0, 1$



$$V = \int_{y=0}^{y=1} \int_{x=y^2}^{x=y} (3x^2 + y^2)\, dx\, dy = \int_{y=0}^{y=1} (x^3 + xy^2)\Big|_{x=y^2}^{x=y} dy$$

$$= \int_{y=0}^{y=1} (y^3 + y^3) - (y^6 + y^4)\ dy = \int_{y=0}^{y=1} (2y^3 - y^6 - y^4)\ dy$$

$$= \left(\frac{y^4}{2} - \frac{y^7}{7} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1} = \left(\frac{1}{2} - \frac{1}{7} - \frac{1}{5}\right) - 0$$

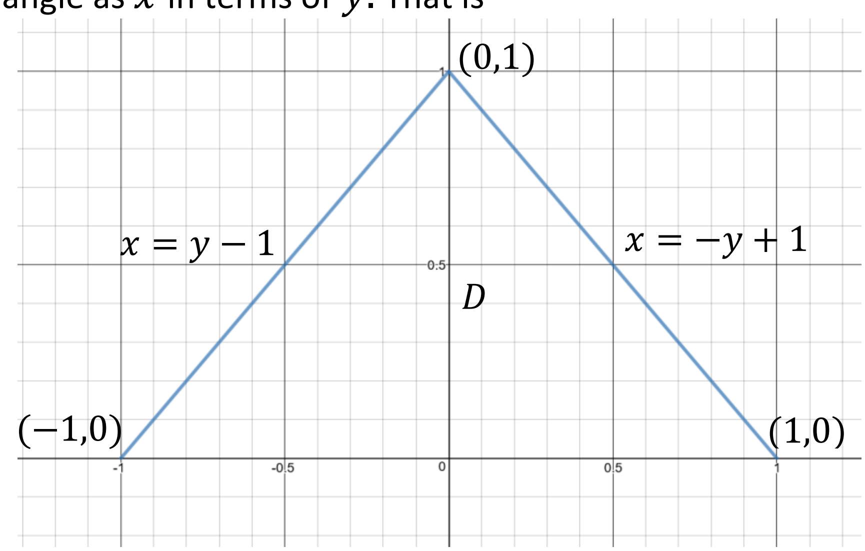
$$= \frac{35}{70} - \frac{10}{70} - \frac{14}{70} = \frac{11}{70}\,.$$

Ex. Evaluate the following integral if the region $D$ is the interior of the triangle whose vertices are $(-1, 0)$, $(1, 0)$, and $(0, 1)$ as an $x$-simple region.

$$\iint_D (x + 2y)\, dA$$

To evaluate this integral as an $x$-simple region we need to express the sides of the triangle as $x$ in terms of $y$. That is

$x = y - 1$
$x = -y + 1$



Notice that we no longer need to break this integral up into 2 integrals as we did when we treated it as a $y$-simple region.

$$\iint_D (x+2y)dA = \int_{y=0}^{y=1}\int_{x=y-1}^{x=-y+1}(x+2y)\,dx\,dy$$

$$= \int_{y=0}^{y=1}\left(\frac{1}{2}x^2+2xy\right)\Big|_{x=y-1}^{x=-y+1}\,dy$$

$$= \int_{y=0}^{y=1}\frac{1}{2}(-y+1)^2+2(-y+1)y-\left[\frac{1}{2}(y-1)^2+2(y-1)y\right]dy$$

Multiplying out and combining terms we get:

$$= \int_{y=0}^{y=1}(-4y^2+4y)dy$$

$$= \left(-\frac{4}{3}y^3+2y^2\right)\Big|_0^1 = -\frac{4}{3}+2 = \frac{2}{3}.$$

Ex. Using a double integral find the area of a circle of radius 2.

Equation of circle of radius 2 is: $x^2+y^2=4.$

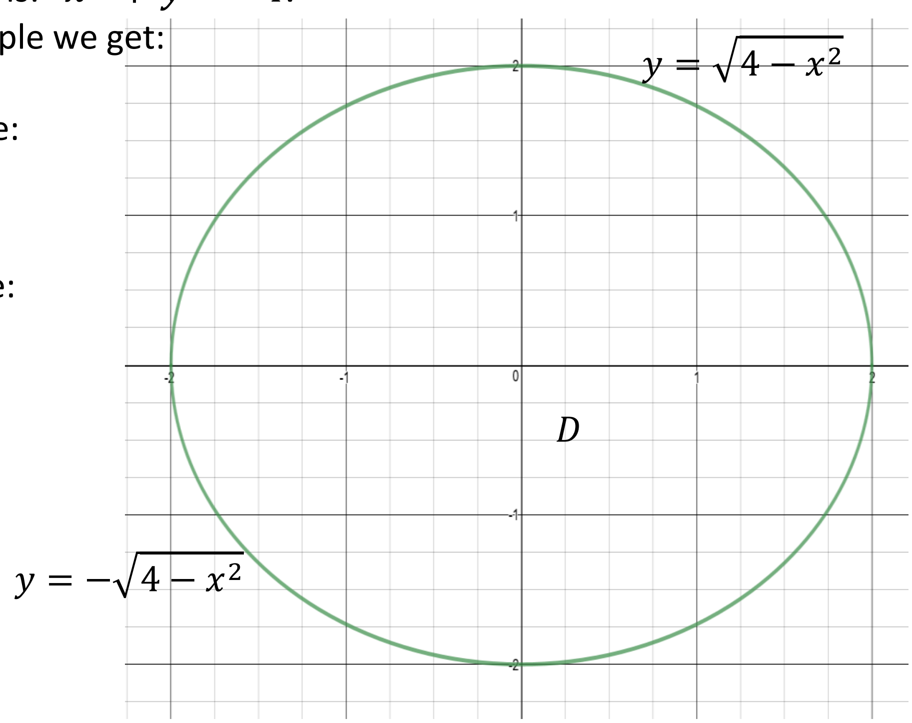
Treating the region as $y$- simple we get:

Equation of upper semi-circle:

$y=\sqrt{4-x^2}.$

Equation of lower semi-circle:

$y=-\sqrt{4-x^2}.$



Area= $\iint_D 1\, dA = \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} 1\, dy\, dx$

$= \int_{x=-2}^{x=2} y|_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$

$= \int_{x=-2}^{x=2} (\sqrt{4-x^2} + \sqrt{4-x^2}) dx$

$= \int_{x=-2}^{x=2} (2\sqrt{4-x^2}) dx$

Now let $x = 2sint$ and $dx = 2(cost)dt$.

When $x = -2,\ t = -\frac{\pi}{2}$, when $x = 2,\ t = \frac{\pi}{2}$.

$= \int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} 2(\sqrt{4 - 4\sin^2 t})\,(2cost)dt$

$= 8\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} (\sqrt{1 - \sin^2 t})\,(cost)dt$

$= 8\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} (\sqrt{\cos^2 t})\,(cost)dt$

$= 8\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} \cos^2 t\, dt$

$= 8\int_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}} (\frac{1}{2} + \frac{1}{2} cos2t)\, dt = 8\,(\frac{1}{2}t + \frac{1}{4} sin2t)\Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}$

$= 8\left[\left(\frac{\pi}{4} + 0\right) - \left(-\frac{\pi}{4} + 0\right)\right] = 4\pi.$