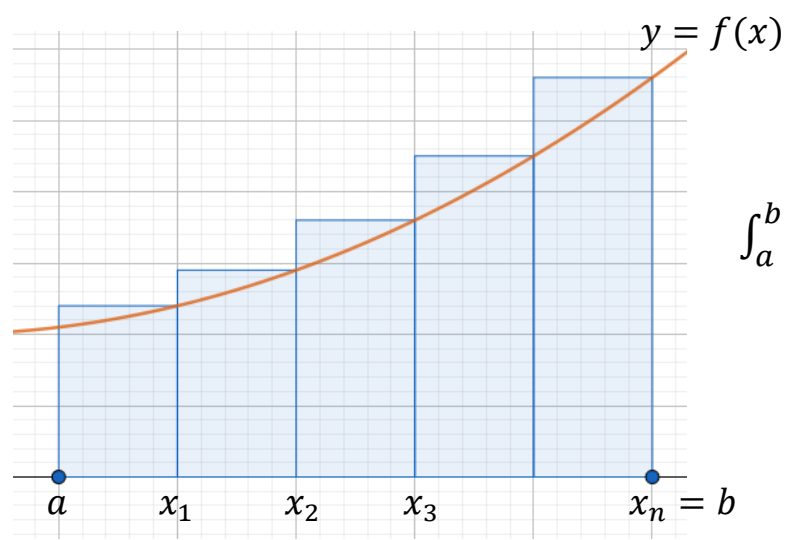


Double Integrals Over Rectangles

Recall for 1 variable: $y = f(x)$



$$\frac{b-a}{n} = \Delta x$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

where $x_i = a + i\Delta x$

$$\int_a^b f(x) dx = \text{area under the curve if } f(x) \geq 0.$$

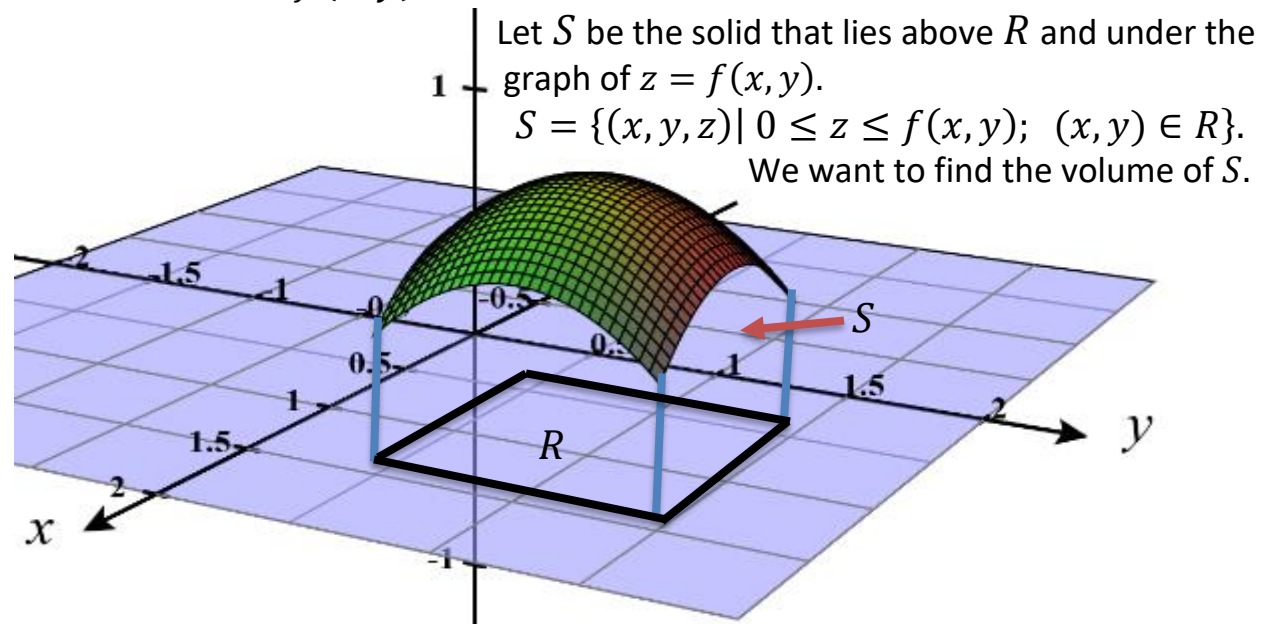
Fundamental Theorem of Calculus:

$$\int_a^b f(x) dx = F(b) - F(a) \text{ if } F'(x) = f(x).$$

For functions of 2 variables we start with a closed rectangle, R :

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

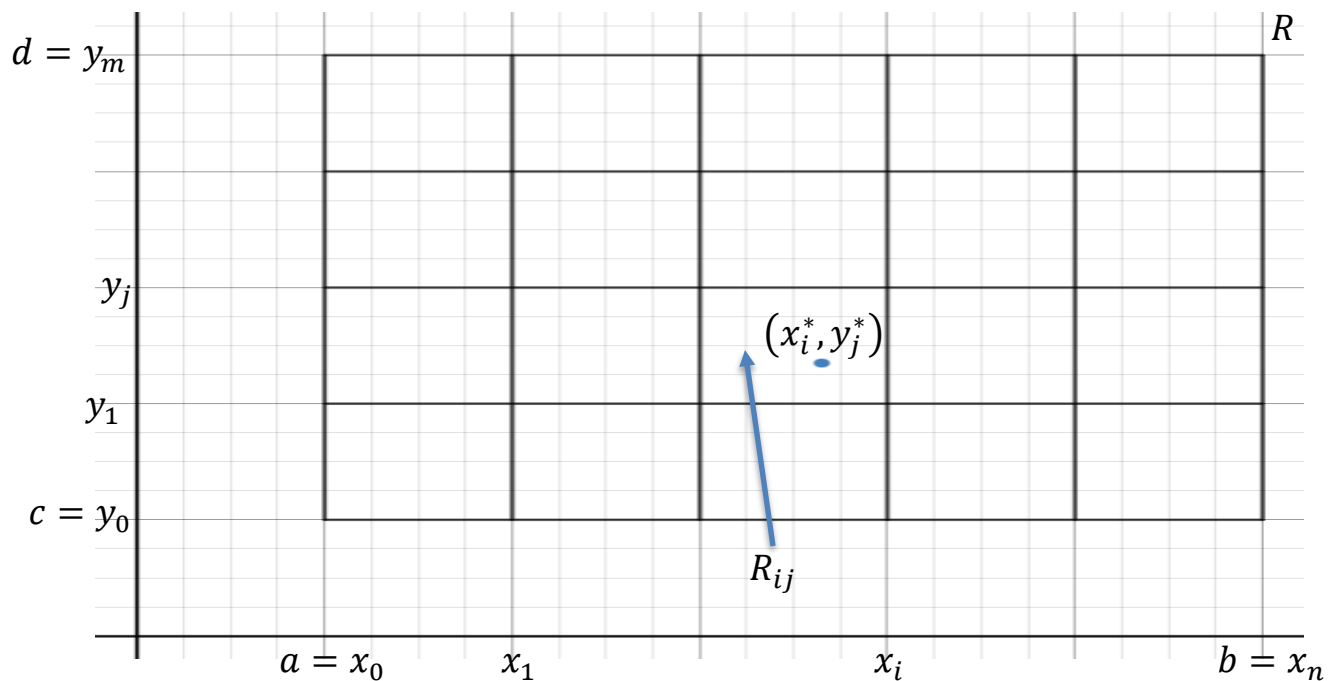
First, let's assume $f(x, y) \geq 0$



First we partition R into subrectangles. We do this by dividing up the intervals $[a, b]$ and $[c, d]$.

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_n = b$$

$$c = y_0 < y_1 < y_2 < \cdots < y_j < \cdots < y_m = d$$

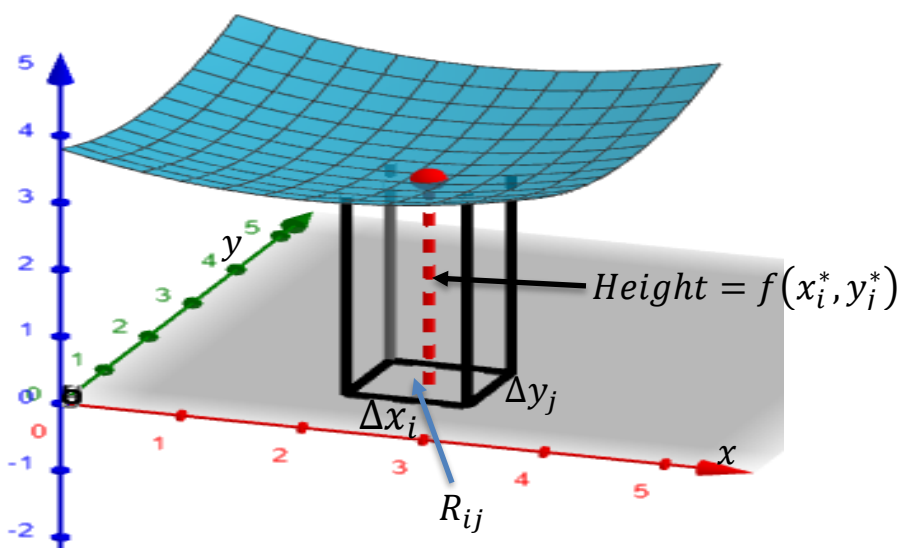


$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

Area of R_{ij} is $\Delta A_{ij} = (\Delta x_i)(\Delta y_j)$

Choose any point in each rectangle R_{ij} and call it (x_i^*, y_j^*) .

The volume of the solid above R_{ij} is approximately $f(x_i^*, y_j^*)\Delta x_i\Delta y_j$.



Adding up the volume of all the solids above the subrectangles, we get:

$$V \approx \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A_{ij}$$

$$\Delta A_{ij} = \Delta x_i \Delta y_j$$

This is called a double Riemann sum.

Define:

$$V = \lim_{\max \Delta x_i, \Delta y_j \rightarrow 0} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*) \Delta A_{ij} = \iint_R f(x, y) dA$$

Note: $f(x, y)$ does not have to be ≥ 0 . If it is, then you get volume, otherwise you don't – much like 1 variable.

$f(x, y)$ is called integrable if the limit exists.

As with 1 dimension, if f is integrable, then we can choose the rectangles, R_{ij} , to be all the same size and we can choose (x_i^*, y_j^*) to be the upper right hand corner.

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \Delta A$$

$$x_i = a + i\Delta x$$

$$y_j = b + j\Delta y$$

$$\Delta A = (\Delta x)(\Delta y).$$

Iterated Integrals

Suppose $f(x, y)$ is continuous on a rectangle, $R = [a, b] \times [c, d]$, by $\int_c^d f(x, y) dy$ we mean to treat x as a constant and integrate $f(x, y)$ in the y variable between c and d .

Ex. Find $\int_{y=0}^{y=3} x^2 + 2y dy$.

$$\begin{aligned} \int_{y=0}^{y=3} x^2 + 2y dy &= x^2 y + y^2 \Big|_{y=0}^{y=3} \\ &= 3x^2 + 9 - (0 + 0) \\ &= 3x^2 + 9. \end{aligned}$$

Notice $\int_c^d f(x, y) dy$ is just a function of x .

If $f(x, y) \geq 0$ then for a fixed x ,

$A(x) = \int_c^d f(x, y) dy$ = cross-sectional area of the solid bounded above by $z = f(x, y)$ and below by R in the x - y plane.

Then $\int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$, called an **iterated integral**,

is the volume of this solid which is equal to $\iint_R f(x, y) dA$ (this is Fubini's theorem).

Ex. Evaluate the iterated integral $\int_1^2 \left[\int_0^3 (x^2 + 2y) dy \right] dx$.

$$\begin{aligned} \int_1^2 \left[\int_0^3 (x^2 + 2y) dy \right] dx &= \int_1^2 3x^2 + 9 dx \\ &= x^3 + 9x \Big|_1^2 \\ &= (8 + 18) - (1 + 9) \\ &= 26 - 10 = 16. \end{aligned}$$

Ex. Evaluate $\int_0^3 \left[\int_1^2 (x^2 + 2y) dx \right] dy$.

$$\begin{aligned} \int_0^3 \left[\int_1^2 (x^2 + 2y) dx \right] dy &= \int_0^3 \frac{1}{3} x^3 + 2xy \Big|_1^2 dy \\ &= \int_0^3 \left[\left(\frac{8}{3} + 4y \right) - \left(\frac{1}{3} + 2y \right) \right] dy \\ &= \int_0^3 \left(\frac{7}{3} + 2y \right) dy \\ &= \frac{7}{3} y + y^2 \Big|_0^3 \\ &= 7 + 9 - 0 = 16. \end{aligned}$$

Notice the two integrals are the same. This is true in general for integrals of continuous functions over rectangles.

Fubini's Theorem: If f is continuous on the rectangle,

$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then:

$$\iint_R f(x, y) dA = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x, y) dy \right] dx = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x, y) dx \right] dy$$

The point is that to integrate over a rectangle we can do this through an iterated integral.

Ex. Evaluate $\iint_R (2xy - 3x^2) dA$ where:

$$R = \{(x, y) \mid 0 \leq x \leq 1, 1 \leq y \leq 3\}.$$

By Fubini's Theorem:

$$\begin{aligned} \iint_R 2xy - 3x^2 dA &= \int_{x=0}^{x=1} \left[\int_{y=1}^{y=3} (2xy - 3x^2) dy \right] dx \\ &= \int_{x=0}^{x=1} xy^2 - 3x^2y \Big|_{y=1}^{y=3} dx \\ &= \int_{x=0}^{x=1} [(9x - 9x^2) - (x - 3x^2)] dx \\ &= \int_{x=0}^{x=1} -6x^2 + 8x dx \\ &= -2x^3 + 4x^2 \Big|_0^1 \\ &= (-2 + 4) - (0 + 0) = 2. \end{aligned}$$

OR

$$\begin{aligned}
\int_{y=1}^{y=3} \int_{x=0}^{x=1} (2xy - 3x^2) dx dy &= \int_{y=1}^{y=3} x^2y - x^3 \Big|_0^1 dy \\
&= \int_{y=1}^{y=3} (y - 1) - (0 - 0) dy \\
&= \int_{y=1}^{y=3} y - 1 dy = \frac{y^2}{2} - y \Big|_1^3 \\
&= \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) \\
&= \frac{3}{2} - \left(-\frac{1}{2}\right) = 2.
\end{aligned}$$

Ex. Find the volume of the solid that lies over the rectangle:

$$R = [0, 3] \times [1, 5] \text{ and below } f(x, y) = 40 - x^2 - y^2.$$

$$\begin{aligned}
V &= \int_{x=0}^{x=3} \left[\int_{y=1}^{y=5} (40 - x^2 - y^2) dy \right] dx \\
&= \int_{x=0}^{x=3} 40y - x^2y - \frac{y^3}{3} \Big|_1^5 dx \\
&= \int_{x=0}^{x=3} \left[\left(200 - 5x^2 - \frac{125}{3}\right) - \left(40 - x^2 - \frac{1}{3}\right) \right] dx \\
&= \int_{x=0}^{x=3} 160 - 4x^2 - \frac{124}{3} dx = \int_{x=0}^{x=3} \frac{356}{3} - 4x^2 dx \\
&= \frac{356}{3}x - \frac{4}{3}x^3 \Big|_0^3 = 356 - 36 = 320.
\end{aligned}$$

Ex. Evaluate $\iint_R ye^{xy} dA$ where $R = [0, 2] \times [0, 1]$.

$$\begin{aligned}\iint_R ye^{xy} dA &= \int_0^1 \int_0^2 ye^{xy} dx dy = \int_0^1 e^{xy} \Big|_0^2 dy \\ &= \int_0^1 e^{2y} - e^0 dy = \int_0^1 e^{2y} - 1 dy = \frac{1}{2} e^{2y} - y \Big|_0^1 \\ &= \left(\frac{1}{2} e^2 - 1 \right) - \left(\frac{1}{2} e^0 - 0 \right) = \frac{1}{2} e^2 - 1 - \frac{1}{2} \\ &= \frac{1}{2} e^2 - \frac{3}{2}.\end{aligned}$$

Note: In the last example it's much easier to integrate with respect to x first. If we integrate with respect to y first, then we would need integration by parts.

Ex. Find the volume of the solid that lies over $R = [0, 1] \times [1, 2]$ and below the surface $z = \frac{3x^2y}{y^2+1}$.

$$\begin{aligned}V &= \int_{x=0}^{x=1} \left[\int_{y=1}^{y=2} \frac{3x^2y}{y^2+1} dy \right] dx \\ &= \int_{x=0}^{x=1} \frac{3}{2} x^2 \ln(y^2 + 1) \Big|_{y=1}^{y=2} dx \\ &= \int_{x=0}^{x=1} \frac{3}{2} x^2 (\ln(5) - \ln(2)) dx \\ &= \int_{x=0}^{x=1} \frac{3}{2} x^2 \ln\left(\frac{5}{2}\right) dx \\ &= \frac{3}{2} \ln\left(\frac{5}{2}\right) \frac{1}{3} x^3 \Big|_{x=0}^{x=1} \\ &= \frac{1}{2} \ln\left(\frac{5}{2}\right).\end{aligned}$$

OR

$$\begin{aligned}
 V &= \int_{y=1}^{y=2} \left[\int_{x=0}^{x=1} \frac{3x^2y}{y^2+1} dx \right] dy \\
 &= \int_{y=1}^{y=2} \frac{x^3y}{y^2+1} \Big|_{x=0}^{x=1} dy \\
 &= \int_{y=1}^{y=2} \frac{y}{y^2+1} dy \\
 &= \frac{1}{2} \ln(y^2 + 1) \Big|_{y=1}^{y=2} \\
 &= \frac{1}{2} (\ln(5) - \ln(2)) = \frac{1}{2} \ln\left(\frac{5}{2}\right).
 \end{aligned}$$

Notice in the special case where $f(x, y) = g(x) h(y)$

(e. g. $f(x, y) = x^3 e^y$):

$$\begin{aligned}
 \iint_R f(x, y) dA &= \int_c^d \int_a^b g(x)h(y) dx dy = \int_c^d h(y) \left[\int_a^b g(x) dx \right] dy \\
 &= \left(\int_c^d h(y) dy \right) \left(\int_a^b g(x) dx \right).
 \end{aligned}$$

Ex. Evaluate $\iint_R 3x^2 e^y dA$; $R = [0, 1] \times [0, \ln 2]$.

$$\begin{aligned}\iint_R 3x^2 e^y dA &= \left(\int_0^1 3x^2 dx \right) \left(\int_0^{\ln 2} e^y dy \right) = (x^3|_0^1)(e^y|_0^{\ln 2}) \\ &= (1 - 0)(e^{\ln 2} - e^0) = 1(2 - 1) = 1.\end{aligned}$$

Properties of double integrals:

1. $\iint_R f(x, y) + g(x, y) dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$

2. $\iint_R c f(x, y) dA = c \iint_R f(x, y) dA$

3. If $f(x, y) \geq g(x, y)$ in R , then:

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA.$$