Vector Fields

Def. A **vector field** in $\mathbb{R}^n$ is a map $\vec{F}: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $x = (x_1, x_2, x_3, \dots, x_n) \in A$, a vector $\vec{F}(x) \in \mathbb{R}^n$.

If $n = 2$, we call $\vec{F}$ a vector field in the plane.

If $n = 3$, we call $\vec{F}$ a vector field in space.

We can always write a vector field in space in the form:

$$\vec{F}(x, y, z) = F_1(x, y, z)\vec{\imath} + F_2(x, y, z)\vec{\jmath} + F_3(x, y, z)\vec{k}$$

or

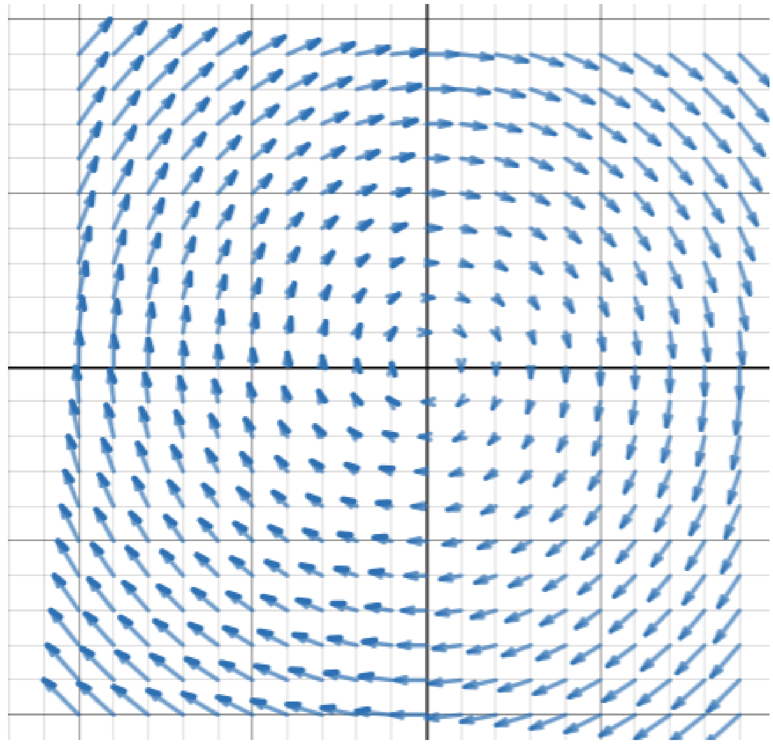
$$\vec{F}(x, y, z) = < F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z) >.$$

Notice that this is different from real-valued functions from $\mathbb{R}^3 \to \mathbb{R}$ (which we will sometimes call a scalar field).

Ex. $\vec{F}(x, y) = 2\vec{\imath} - 3\vec{\jmath} = < 2, -3 >$ is a vector field in the plane drawn as

Ex. $\vec{F}(x,y) = y\vec{\imath} - x\vec{\jmath} = < y, -x >$ is a vector field in the plane

Point	$\vec{F}(x,y)$
$(1,0)$	$< 0, -1 >$
$(1,-1)$	$< -1, -1 >$
$(0,-2)$	$< -2,0 >$
$(-2,-2)$	$< -2,2 >$
$(-4,0)$	$< 0,4 >$
$(-4,4)$	$< 4,4 >$
$(0,8)$	$< 8,0 >$
$(8,8)$	$< 8,-8 >$



Ex. $\vec{F}(x,y,z) = (x^2z)\vec{\imath} + e^y\vec{\jmath} + \sin(xz)\vec{k}$ is a vector field on $\mathbb{R}^3$.

$f(x,y,z) = x^2z + e^y + \sin(xz)$ is a real-valued function on $\mathbb{R}^3$.

Notice that for every value of x, y, z, $\vec{F}(x,y,z)$ gives us a vector in $\mathbb{R}^3$.

For every value of x, y, z, $f(x,y,z)$ gives us a real number, not a vector in $\mathbb{R}^3$.

Ex. A mass, M, at the origin in $\mathbb{R}^3$ exerts a force on a mass, m, located at $\vec{r} = < x, y, z >$ with a magnitude of $\frac{GmM}{|\vec{r}|^2}$, where G is a gravitational constant and the direction is toward the origin. Thus, we can write the force field as:

$$\vec{F}(x,y,z) = \left(\frac{GmM}{|\vec{r}|^2}\right)\left(-\frac{\vec{r}}{|\vec{r}|}\right) = -\left(\frac{GmM}{|\vec{r}|^3}\right)\vec{r}\ ,$$

$$\frac{\vec{r}}{|\vec{r}|^3} = < \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} > .$$

So we can write $\vec{F}(x,y,z)$ as:

$$\vec{F}(x,y,z) = < \frac{-GmMx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-GmMy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-GmMz}{(x^2+y^2+z^2)^{\frac{3}{2}}} >.$$

The gradient of a real-valued function is a vector field.

Def. If $f: \mathbb{R}^3 \to \mathbb{R}$, then the **gradient** of a f, ∇f, is defined to be:

$$\nabla f(x,y,z) = < f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) >$$
$$= (f_x)\vec{\imath} + (f_y)\vec{\jmath} + (f_z)\vec{k}$$

Ex. $f(x,y,z) = xe^{yz} + z^2$ find the vector field ∇f.

$$\nabla f = \frac{\partial}{\partial x}(xe^{yz} + z^2)\vec{\imath} + \frac{\partial}{\partial y}(xe^{yz} + z^2)\vec{\jmath} + \frac{\partial}{\partial z}(xe^{yz} + z^2)\vec{k}$$

$$= (e^{yz})\vec{\imath} + (xze^{yz})\vec{\jmath} + (xye^{yz} + 2z)\vec{k}.$$

Def. A vector field, $\vec{V}$, that is the gradient of a function $f: \mathbb{R}^n \to \mathbb{R}$, i.e. $\vec{V} = \nabla f$, is called a **gradient vector field**.

Not all vector fields are gradient vector fields. However, the ones that are gradient vector fields turn out to have special properties (which you will see if you take vector analysis). So given a vector field we might ask if it's a gradient vector field.

Ex. Show that the vector field $\vec{V}(x, y) = y^2\vec{\imath} - x^2\vec{\jmath}$ is not a gradient vector field.

To be a gradient vector field $\vec{V} = \nabla f$ for some function f:

$$\vec{V}(x, y) = y^2\vec{\imath} - x^2\vec{\jmath} = f_x\vec{\imath} + f_y\vec{\jmath}.$$

But if $f_x = y^2$ and $f_y = -x^2$, we would need $f_{xy} = f_{yx}$ since f_x, f_y have continuous derivatives.

However, $f_{xy} = 2y$ and $f_{yx} = -2x$, which are only equal at $(0, 0)$
So $\vec{V} \neq \nabla f$.

Ex. Show that the vector field $\vec{F}(x, y, z) = < x, y, z^2 >$ is a gradient vector field.

We need to find a function $f: \mathbb{R}^3 \to \mathbb{R}$ such that:

$$\nabla f = < f_x, f_y, f_z > = < x, y, z^2 >$$

In other words:

$$f_x = x; \qquad f_y = y; \qquad f_z = z^2.$$

Since this is a relatively simple set of partial differential equation, we can "guess" an answer that works:

$$f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{3}z^3.$$

It's easy to check that $\nabla f = \vec{F}(x,y,z) = < x, y, z^2 >$ and so $\vec{F}(x,y,z) = < x, y, z^2 >$ is a gradient vector field.

Ex. The vector field (called the gravitational vector field) given by:

$$\vec{F}(x,y,z) = < \frac{-GmMx}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-GmMy}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{-GmMz}{(x^2+y^2+z^2)^{\frac{3}{2}}} >$$

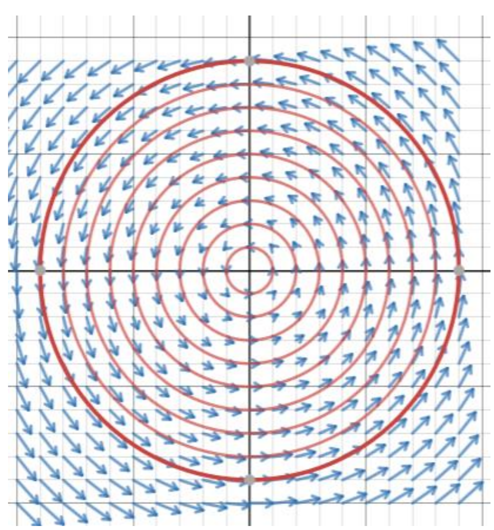
is a gradient vector field since if $f(x,y,z) = \frac{mMG}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

then, $\nabla f = \vec{F}$.

<u>Flow Lines</u>

Def. If $\vec{F}$ is a vector field, a **flow line** for $\vec{F}$ is a path $c(t)$ such that:

$$c'(t) = \vec{F}(c(t)).$$



Flow lines show up in the study of differential equations.

If $\quad c(t) = < x(t), y(t), z(t) >$ and

$\vec{F}(x, y, z) = < P(x, y, z), Q(x, y, z), R(x, y, z) >$, then:

$c'(t) = < x'(t), y'(t), z'(t) >$ and

$\vec{F}\big(c(t)\big) = < P\big(x(t), y(t), z(t)\big), Q\big(x(t), y(t), z(t)\big), R\big(x(t), y(t), z(t)\big) >$.

So a flow line is a solution to the system of differential equations:

$$c'(t) = \vec{F}\big(c(t)\big)$$

$$\begin{aligned} x'(t) &= P\big(x(t), y(t), z(t)\big) \\ y'(t) &= Q\big(x(t), y(t), z(t)\big) \\ z'(t) &= R\big(x(t), y(t), z(t)\big). \end{aligned}$$

Ex. Show that $c(t) = < e^{2t}, \log|t|, \frac{1}{t} >, t \neq 0$, is a flow line for:

$$\vec{F}(x, y, z) = < 2x, z, -z^2 >.$$

We need to show that $c'(t) = \vec{F}\big(c(t)\big)$:

$$c'(t) = < 2e^{2t}, \frac{1}{t}, -\frac{1}{t^2} >$$

$$\vec{F}\big(c(t)\big) = < 2e^{2t}, \frac{1}{t}, -\frac{1}{t^2} >$$

So $c'(t) = \vec{F}\big(c(t)\big)$ and $c(t)$ is a flow line for $\vec{F}$.

Ex. Show that $c(t) =< sint, cost, e^t >$ is a flow line for:

$$\vec{F}(x, y, z) = < y, -x, z >.$$

We need to show that $c'(t) = \vec{F}\big(c(t)\big)$:

$$c'(t) = < cost, -sint,\ e^t >$$

$$\vec{F}\big(c(t)\big) = < cost, -sint,\ e^t >$$

So $c'(t) = \vec{F}\big(c(t)\big)$ and $c(t)$ is a flow line for $\vec{F}$.