Newton's Second Law and Circular Motion

A path or curve is a map $c: \mathbb{R} \to \mathbb{R}^n$ or $c: I \subseteq \mathbb{R} \to \mathbb{R}^n$, where I is an interval.

$$c'(t) = v(t) = \text{velocity vector}$$
$$c''(t) = a(t) = \text{acceleration vector}$$
$$\|v(t)\| = \text{speed}$$

Ex. Let $c(t) = < 2\cos t, 2\sin t, 4t >$. Find the velocity and acceleration vectors at $t = \frac{\pi}{4}$, and the speed.

$$v(t) = c'(t) = < -2\sin t, 2\cos t, 4 >$$

$$v\left(\frac{\pi}{4}\right) = < -2\sin\frac{\pi}{4}, 2\cos\frac{\pi}{4}, 4 > = < -\sqrt{2}, \sqrt{2}, 4 >$$

$$a(t) = < -2\cos t, -2\sin t, 0 >$$

$$a\left(\frac{\pi}{4}\right) = < -\sqrt{2}, -\sqrt{2}, 0 >.$$

$$\text{Speed} = \left\|v\left(\frac{\pi}{4}\right)\right\| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}.$$

A curve in $\mathbb{R}^3$ has the form:

$$c(t) = < x(t), y(t), z(t) >.$$

Thus the velocity and acceleration vectors are:

$$v(t) = < x'(t), y'(t), z'(t) >$$
$$a(t) = < x''(t), y''(t), z''(t) >.$$

Def. A differentiable path, c, is said to be **regular** at $t = t_0$ if $c'(t_0) \neq \vec{0}$. If $c'(t) \neq \vec{0}$ for all t, then we say c is a **regular path.**

Ex. Where is the path $c(t) = < t^2, cost\ , e^{t^2} >$ regular?

$$c'(t) = < 2t, -sint\ , 2te^{t^2} >$$

$c'(t) = \vec{0}$ only when $t = 0$

So $c(t)$ is regular when $t \neq 0$.

Ex. The acceleration, initial velocity, and initial position of a particle traveling through space are given by:

$$a(t) = < 2, -6, -4 >$$
$$v(0) = < -5, 1, 3 >$$
$$r(0) = < 6, -2, -1 >$$

The particle's trajectory (path), $r(t)$, intersects the yz plane exactly twice. Find the intersection points.

$a(t) = < x''(t), y''(t), z''(t) > = < 2, -6, -4 >$; thus by integration:

$$x'(t) = 2t + c_1$$
$$y'(t) = -6t + c_2$$
$$z'(t) = -4t + c_3$$

$v(0) = < x'(0), y'(0), z'(0) > = < -5, 1, 3 >$; thus we have:

$$-5 = x'(0) = 2(0) + c_1 \quad \Rightarrow \quad c_1 = -5$$
$$1 = y'(0) = -6(0) + c_2 \quad \Rightarrow \quad c_2 = 1$$
$$3 = z'(0) = -4(0) + c_3 \quad \Rightarrow \quad c_3 = 3$$

Thus, $v(t) = < 2t - 5,\ -6t + 1,\ -4t + 3 > = < x'(t), y'(t), z'(t) >$.

Integrating again we get:

$$\begin{aligned} x(t) &= \quad t^2 - 5t + d_1 \\ y(t) &= -3t^2 + \ \ t + d_2 \\ z(t) &= -2t^2 + 3t + d_3 \end{aligned}$$

$r(0) = < x(0), y(0), z(0) > = < 6, -2, -1 >$; Thus we have:

$$\begin{aligned} 6 = x(0) &= \quad 0^2 - 5(0) + d_1 & \Rightarrow \quad 6 &= d_1 \\ -2 = y(0) &= -3(0)^2 + \quad 0 + d_2 & \Rightarrow \quad -2 &= d_2 \\ -1 = z(0) &= -2(0)^2 + 3(0) + d_3 & \Rightarrow \quad -1 &= d_3 \end{aligned}$$

So we get:

$$r(t) = < (t^2 - 5t + 6),\ (-3t^2 + t - 2),\ (-2t^2 + 3t - 1) >.$$

$r(t)$ intersects the yz plane when the x component is 0.

$$\begin{gathered} t^2 - 5t + 6 = 0 \\ (t-3)(t-2) = 0 \\ t = 2, 3 \end{gathered}$$

$$\begin{aligned} t = 2: &\quad r(2) = < 0, -12, -3 > \\ t = 3: &\quad r(3) = < 0, -26, -10 > \end{aligned}$$

These are the points where $r(t)$ intersects with the yz plane.

Newton's Second Law: $F = ma = mc''(t)$

Suppose a mass, m, is moving in a circular path of radius r_0 at a constant speed, s. We can express this as:

$$r(t) = < r_0 \cos(\frac{st}{r_0}), r_0 \sin(\frac{st}{r_0}) >$$

Notice that:

$$v(t) = r'(t) = < -s \sin\left(\frac{st}{r_0}\right), s \cos\left(\frac{st}{r_0}\right) >$$

$$\text{speed} = \|v(t)\| = \sqrt{s^2\left(\sin^2\left(\frac{st}{r_0}\right)\right) + s^2\left(\cos^2\left(\frac{st}{r_0}\right)\right)} = s.$$

The quantity $\frac{s}{r_0}$ is called the **frequency**, w. Thus we can write:

$$r(t) = < r_0 \cos(wt), r_0 \sin(wt) >.$$

So we have:

$$v(t) = r'(t) =< -r_0 w\big(sin(wt)\big), r_0 w\,(\cos(wt)) >.$$

Calculating the acceleration we get:

$$a(t) = v'(t) = < -r_0 w^2 \cos(wt), -r_o w^2 \sin(wt) >$$
$$= -w^2\big(r(t)\big).$$

Thus when we have circular motion, the acceleration is in the opposite direction to $r(t)$ (towards the center). This acceleration multiplied by the mass is called the **centripetal force**.

Ex. A body of mass 3 kilograms moves on a circle of radius 4 meters, making one revolution every 6 seconds. Find the centripetal force on the body.

We know $a = -w^2\big(r(t)\big)$ where:

$$r(t) = < r_0 \cos(wt), r_0 \sin(wt) >.$$

$$F = ma = m(-w^2) < r_0 \cos(wt), r_0 \sin(wt) >,$$

where $w = \frac{s}{r_0}$.

Since the speed, s, is given by: $s = \frac{2\pi(4)}{6} = \frac{4\pi}{3}$,

We have:

$$w = \frac{s}{r_0} = \frac{4\pi}{3}\left(\frac{1}{4}\right) = \frac{\pi}{3} \text{ and}$$

$$m = 3.$$

Thus the centripetal force is given by:

$$F = m(-w^2) < r_0 \cos(wt), r_0 \sin(wt) >$$

$$F = 3\left(-\left(\frac{\pi}{3}\right)^2\right) < 4\cos\left(\frac{\pi t}{3}\right), 4\sin\left(\frac{\pi t}{3}\right) >$$

$$F = -\frac{4\pi^2}{3} < \cos\left(\frac{\pi t}{3}\right), \sin\left(\frac{\pi t}{3}\right) >.$$