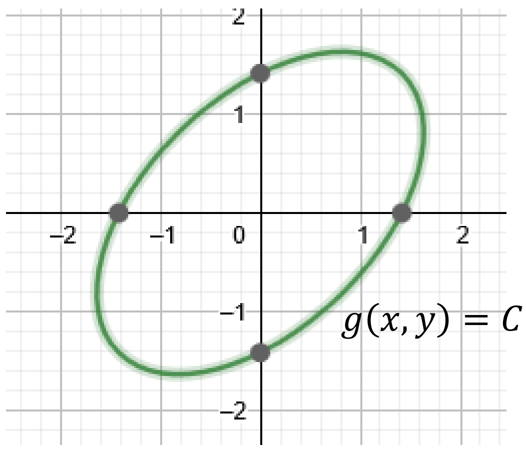
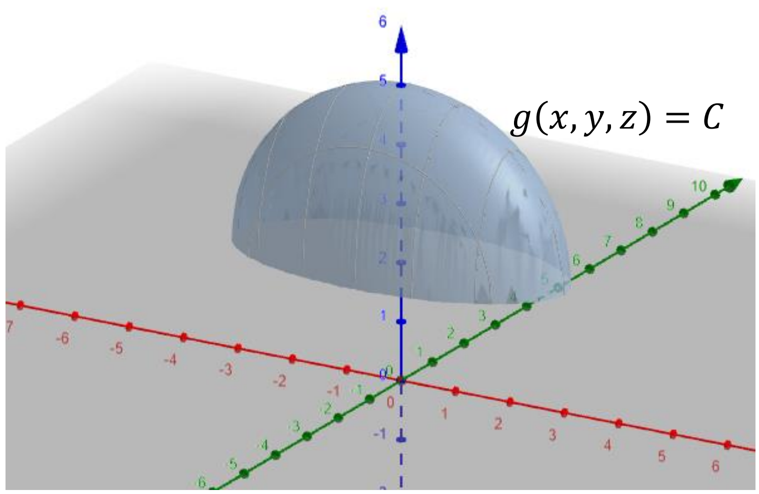
Lagrange Multipliers

In the previous section, we saw how to find the absolute maximum and minimum of a real-valued function, $f(x, y)$, on a bounded domained, $D \subseteq \mathbb{R}^2$, where the boundary of D is a curve we can parametrize. Now we want to be able to find the absolute maximum and minimum of a real-valued function, $f(x, y)$, on a general smooth curve in $\mathbb{R}^2$ given by $g(x, y) = C$.



In addition, we would like to be able to find the absolute maximum and minimum of a real-valued function, $f(x, y, z)$, on a general smooth surface in $\mathbb{R}^3$ given by $g(x, y, z) = C$.



For example, suppose we want to know the maximum value of $f(x, y, z) = x + z$ subject to the constraint that (x, y, z) must lie on the unit sphere, $x^2 + y^2 + z^2 = 1$. Let's call this constraint set S.

Notice that even for functions of 1 variable, a continuous function need not have a maximum or minimum value. For example, $f(x) = x$ doesn't have a maximum or minimum value if $x \in \mathbb{R}$ or if $0 < x < 1$. However, if the constraint set S is closed and bounded, and f is a continuous function, then f does have a maximum and a minimum value on S.

We know for a real-valued function, $f: \mathbb{R}^3 \to \mathbb{R}$, we search for relative maxima and minima at points where $\nabla f(x_0, y_0, z_0) = 0$, where $(x_0, y_0, z_0) \in \mathbb{R}^3$. These are critical points. Now, we want to find maxima and minima for $f(x, y, z)$ when its domain is restricted to a level surface, S, in $\mathbb{R}^3$ given by $g(x, y, z) = C$.

Theorem (The Method of Lagrange Multipliers):
Suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{R}^n \to \mathbb{R}$, where $n = 2$ or 3 are continuously differentiable functions. Let S be the level surface (or curve) given by $g(x, y, z) = C$. Assume $\nabla g(x, y, z) \neq \vec{0}$. If f restricted to S has a local maximum or minimum at (x_0, y_0, z_0), then there is a real number, λ (which might be 0), such that:

$$\nabla f(x_0, y_0, z_0) = \lambda\big(\nabla g(x_0, y_0, z_0)\big)$$

In this case, we say (x_0, y_0, z_0) is a critical point of f restricted to S.

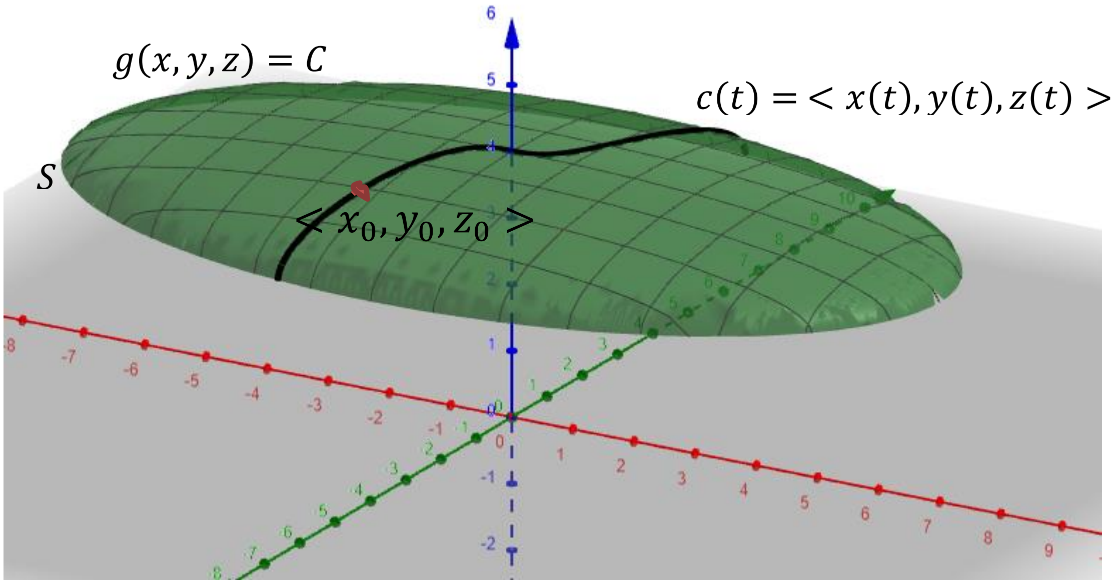
Proof: We have already seen that $\nabla g(x, y, z)$ is perpendicular to the tangent plane of S at $(x, y, z) \in S$, since:

$$S = \{(x, y, z) \in \mathbb{R}^3 |\ g(x, y, z) = C\}.$$

This is true for any point $(x, y, z) \in S$.

Now, let's show if $f: \mathbb{R}^3 \to \mathbb{R}$ has a relative maximum or minimum at $(x_0, y_0, z_0) \in S$, then $\nabla f(x_0, y_0, z_0)$ is also perpendicular to the tangent plane to S at (x_0, y_0, z_0).

Let $c(t) = < x(t), y(t), z(t) >$ be any smooth curve on S where $c(0) = < x(0), y(0), z(0) > = < x_0, y_0, z_0 >$ is a relative maximum or minimum of f restricted to S.



Thus, $f(c(t)) = f(x(t), y(t), z(t))$ is a function of one variable, t.

By the Chain Rule:

$$\frac{d}{dt}\Big(f\big(x(t), y(t), z(t)\big)\Big) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \nabla f \cdot < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > = \nabla f \cdot c'(t).$$

From one variable calculus we know if f is differentiable and $t = 0$ is a relative maximum or minimum, then $\frac{d}{dt}(f) = 0$, when $t = 0$.

So at a relative maximum or minimum:

$$0 = \nabla f\big(x(0), y(0), z(0)\big) \cdot c'(0).$$

Thus, $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent vector of every smooth curve on S that goes through (x_0, y_0, z_0).

So ∇f is perpendicular to the tangent plane of S at (x_0, y_0, z_0). Hence:

$$\nabla f(x_0, y_0, z_0) = \lambda\big(\nabla g(x_0, y_0, z_0)\big).$$

λ is known as the **Lagrange multiplier** for f and g.

So when given a real-valued function $f(x, y, z)$ (or $f(x, y)$) and a constraint set $g(x, y, z) = C$ (or $g(x, y) = C$), to find relative maxima and minima of $f(x, y, z)$ restricted to S, given by $g(x, y, z) = C$ we solve for all points (x, y, z) such that:

$$\nabla f(x, y, z) = \lambda\big(\nabla g(x, y, z)\big)$$

1. $\frac{\partial f}{\partial x} = \lambda\left(\frac{\partial g}{\partial x}\right)$
2. $\frac{\partial f}{\partial y} = \lambda\left(\frac{\partial g}{\partial y}\right)$
3. $\frac{\partial f}{\partial z} = \lambda\left(\frac{\partial g}{\partial z}\right)$
4. $g(x, y, z) = C$.

So if $f: \mathbb{R}^3 \to \mathbb{R}$ and $g(x, y, z) = C$ is the constraint, then we will have to solve 4 equations in 4 unknowns (x, y, z, λ).

If $f: \mathbb{R}^2 \to \mathbb{R}$ and $g(x, y) = C$ is the constraint, then we will have to solve 3 equations in 3 unknowns (x, y, λ).

Ex. Maximize $f(x, y, z) = y + z$ subject to $x^2 + y^z + z^2 = 1$.

Here, the constraint set, S, is the unit sphere:

$$g(x, y, z) = x^2 + y^z + z^2 = 1.$$

Since $f(x, y, z) = y + z$ is continuous and S is closed and bounded, there will be absolute maximum and minimum values of f on S.

$$\nabla f = < f_x, f_y, f_z > = < 0, 1, 1 >$$

$$\nabla g = < 2x, 2y, 2z >$$

We get 3 equations from $\nabla f = \lambda(\nabla g)$:

1. $0 = 2\lambda x$
2. $1 = 2\lambda y$
3. $1 = 2\lambda z$

And we get a 4th equation from the constraint:

4. $x^2 + y^z + z^2 = 1.$

From equation 2 ($1 = 2\lambda y$), we know $\lambda \neq 0$.

Thus, from equation 1 ($0 = 2\lambda x$), we can conclude $x = 0$.

From equations 2 and 3 we have:

$$2\lambda y = 2\lambda z.$$

Since $\lambda \neq 0$, this means:

$$y = z.$$

Now use equation 4 ($x^2 + y^z + z^2 = 1$) and the fact that we know $x = 0$ and $y = z$ to get:

$$(0)^2 + y^2 + z^2 = 1$$

$$2y^2 = 1$$

$$y = \pm\frac{1}{\sqrt{2}}.$$

Notice: we never found λ, which is okay since we really want all points (x, y, z) that satisfy the 4 equations.

Thus, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are our critical points:

$$f\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

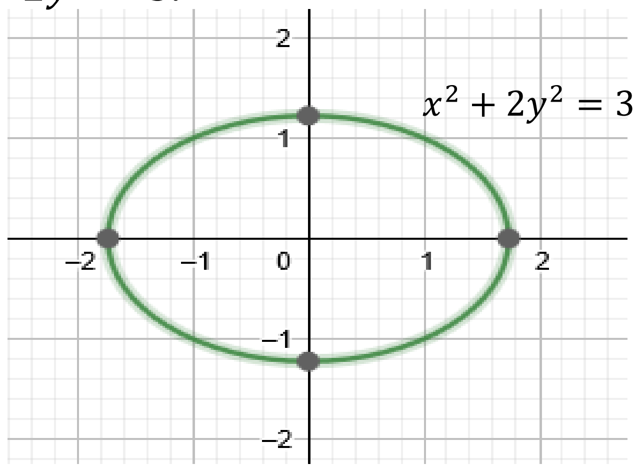
$$f\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$

So, the absolute maximum value of f restricted to $x^2 + y^z + z^2 = 1$ is $\sqrt{2}$ (at the point $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$)

The absolute minimum value of f restricted to $x^2 + y^z + z^2 = 1$ is $-\sqrt{2}$ (at the point $\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$).

Ex. Find the extrema (maxima and minima) of $f(x, y) = x$ subject to $x^2 + 2y^2 = 3$.

In this example, $f: \mathbb{R}^2 \to \mathbb{R}$ is restricted to S, an ellipse, so we can write: $g(x, y) = x^2 + 2y^2 = 3$.



Since $f: \mathbb{R}^2 \to \mathbb{R}$ we will get 2 equations from $\nabla f = \lambda(\nabla g)$ plus 1 equation from the constraint, $x^2 + 2y^2 = 3$.

$$\nabla f = < f_x, f_y > = < 1, 0 >$$

$$\nabla g = < g_x, g_y > = < 2x, 4y >$$

$$\nabla f = \lambda(\nabla g)$$

1. $1 = 2\lambda x$
2. $0 = 4\lambda y$
3. $x^2 + 2y^2 = 3$

From equation 1, we know $\lambda \neq 0$. Thus, from equation 2, $y = 0$. Plugging in $y = 0$ into equation 3 we get:

$$x^2 = 3 \;\Rightarrow\; x = \pm\sqrt{3}.$$

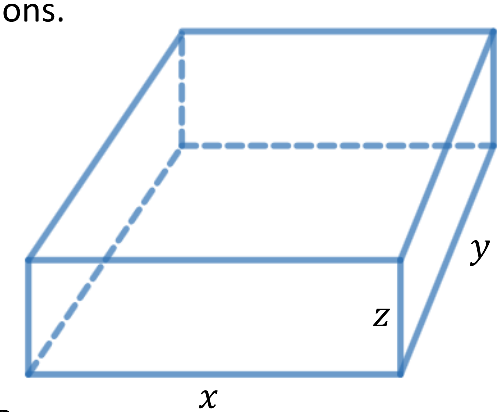
Thus, the critical points are $(\sqrt{3}, 0), (-\sqrt{3}, 0)$.

Notice: once again we don't need to find λ.

$$f(\sqrt{3},0) = \sqrt{3}$$
$$f(-\sqrt{3},0) = -\sqrt{3}.$$

Since S, an ellipse, is closed and bounded and $f(x,y) = x$ is continuous on S, f must take on its absolute maximum and minimum values. Thus, the absolute maximum value of f on S is $\sqrt{3}$ and the absolute minimum value of f on S is $-\sqrt{3}$.

Ex. Assume that among all rectangular boxes with a surface area of $54m^2$ there is a box of largest possible volume. Find its dimensions.



We want to maximize the volume:

$$f(x,y,z) = xyz$$

Subject to:

$$g(x,y,z) = 2xy + 2xz + 2yz = 54.$$

$$\nabla f = < yz, xz, xy >$$

$$\nabla g = < 2y + 2z,\ 2x + 2z,\ 2x + 2y >$$

So $\nabla f = \lambda(\nabla g)$ gives us the following equations:

1. $yz = 2\lambda(y + z)$
2. $xz = 2\lambda(x + z)$
3. $xy = 2\lambda(x + y)$

The constraint gives the 4th equation:

4. $2xy + 2xz + 2yz = 54$ i.e. $xy + xz + yz = 27.$

Notice that $x \neq 0, y \neq 0, z \neq 0$, otherwise $f(x,y,z) = 0$.

Solve equations 1, 2, and 3 for 2λ:

1. $\frac{yz}{y+z} = 2\lambda$
2. $\frac{xz}{x+z} = 2\lambda$
3. $\frac{xy}{x+y} = 2\lambda$

Setting equations 1 and 2 equal to each other:

$$\frac{yz}{y+z} = \frac{xz}{x+z}$$
$$(x+z)(yz) = xz(y+z)$$
$$yz^2 = xz^2$$
$$y = x \quad \text{(since } z \neq 0\text{).}$$

Similarly, setting equations 2 and 3 equal to each other:

$$\frac{xz}{x+z} = \frac{xy}{x+y}$$
$$(x+y)(xz) = (xy)(x+z)$$
$$x^2z = x^2y$$
$$z = y \quad \text{(since } x \neq 0\text{).}$$

Thus $x = y = z$. Now plug into equation 4:

$$xy + xz + yz = 27$$
$$x^2 + x^2 + x^2 = 27$$
$$3x^2 = 27$$
$$x = \pm 3.$$

So the dimensions that maximize the volume are:

$$3m \times 3m \times 3m.$$

Global Maxima and Minima on Bounded Regions

Def. Let $U \subseteq \mathbb{R}^n, n = 2$ or 3, U is open with a boundary ∂U. We say $\boldsymbol{\partial U}$ **is smooth** if ∂U is the level set of a smooth function, g, whose gradient $\nabla g \neq 0$.

Lagrange Multiplier Strategy for Finding Absolute Maxima and Minima on Bounded Regions, U:

1. Locate all critical points of f in U (i.e. $\nabla f = 0$)
2. Use Lagrange multipliers to find all critical points of f on ∂U (i.e. $\nabla f = \lambda(\nabla g)$)
3. Compute the values of f at all of the critical points
4. Select the largest and smallest values

Ex. Find the absolute maximum and minimum values of

$$f(x, y, z) = x^2 + y^2 + z^2 + x - z$$

on the set:

$$D = \{(x, y, z) | \, x^2 + y^2 + z^2 \leq 1\}.$$

Here $D = U \cup (\partial U)$ where:

$$U = \{(x, y, z) | \, x^2 + y^2 + z^2 < 1\}$$

$$\partial U = \{(x, y, z) | \, x^2 + y^2 + z^2 = 1\}.$$

1. Start by finding all critical points of f in U:

$$\nabla f = < f_x, f_y, f_x > = < 2x + 1,\ \ 2y,\ \ 2z - 1 >$$

$$f_x = 2x + 1 = 0 \qquad \Rightarrow \qquad x = -\frac{1}{2}$$

$$f_y = 2y = 0 \qquad \Rightarrow \qquad y = 0$$

$$f_z = 2z - 1 = 0 \qquad \Rightarrow \qquad z = \frac{1}{2}$$

Critical point in U: $(-\frac{1}{2}, 0, \frac{1}{2})$.

2. Find all critical points on $\partial U = \{(x, y, z)|\ x^2 + y^2 + z^2 = 1\}$.

$$g(x, y, z) = x^2 + y^2 + z^2 = 1$$

$$\nabla g = < g_x, g_y, g_z > = < 2x, 2y, 2z >$$

$\nabla f = \lambda(\nabla g)$:

1. $2x + 1 = 2\lambda x$
2. $2y = 2\lambda y$
3. $2z - 1 = 2\lambda z$
4. $x^2 + y^2 + z^2 = 1.$

Notice $\lambda \neq 1$ since if $\lambda = 1$, then by equation 1:

$2x + 1 = 2x$, which has no solution.

Since $\lambda \neq 1$, by equation 2:

$$2y = 2\lambda y \implies y = 0.$$

Plugging into equation 4, we get:

$$x^2 + z^2 = 1.$$

Now solve equations 1 and 3 for 2λ:

1. $\frac{2x+1}{x} = 2\lambda$
2. $\frac{2z-1}{z} = 2\lambda$

Setting these equations equal to each other:

$$\frac{2x+1}{x} = \frac{2z-1}{z}$$
$$z(2x+1) = (2z-1)x$$
$$z = -x.$$

Now plug this into equation 4 with $y = 0$:

$$x^2 + z^2 = 1$$
$$x^2 + (-x)^2 = 1$$
$$2x^2 = 1$$
$$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}.$$

So the critical points on ∂U are:

$$\left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right); \qquad \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right).$$

3. Now find the values of f at all of the critical points.

$$f\left(-\frac{1}{2}, 0, \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$$

$$f\left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$$

$$f\left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 - \frac{2}{\sqrt{2}} = 1 - \sqrt{2}.$$

The absolute maximum of f is $1+\sqrt{2}$ (at the point $\left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$).

The absolute minimum of f is $-\frac{1}{2}$ (at the point $\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$).

Ex. Find the points where $f(x,y) = x^2 + xy + y^2$ subject to $x^2 + y^2 \leq 1$ attains its maximum and minimum values. Find those values.

1. Find the critical points of $f(x,y) = x^2 + xy + y^2$ for $\{(x,y) |\ x^2 + y^2 < 1\}$.

   $f_x = 2x + y = 0 \Longrightarrow \quad y = -2x$

   $f_y = 2y + x = 0 \Longrightarrow \quad 2(-2x) + x = 0 \Longrightarrow \quad x = 0,\ y = 0.$

   Only critical point is $(0,0)$.

2. Find all critical points on the boundary, $\{(x,y) |\ x^2 + y^2 = 1\}$.

   So $g(x,y) = x^2 + y^2 = 1.$

   $\nabla g = < g_x, g_y > = < 2x, 2y >$

   $\nabla f = < f_x, f_y > = < 2x + y, 2y + x >.$

$\nabla f = \lambda(\nabla g)$:

1. $2x + y = 2\lambda x$
2. $2y + x = 2\lambda y$
3. $x^2 + y^2 = 1$

$x \neq 0$, since from equation 1, if $x = 0$ then $y = 0$, but that doesn't satisfy equation 3, $x^2 + y^2 = 1$.

$y \neq 0$, since from equation 2, if $y = 0$ then $x = 0$.

Now solve equations 1 and 2 for 2λ:

1. $\frac{2x+y}{x} = 2\lambda$
2. $\frac{2y+x}{y} = 2\lambda$.

Setting the expressions for 2λ equal to each other:

$$\frac{2x+y}{x} = \frac{2y+x}{y}$$
$$y(2x+y) = x(2y+x)$$
$$2xy + y^2 = 2xy + x^2$$
$$y^2 = x^2$$
$$y = \pm x.$$

Now plug $y = \pm x$ into $x^2 + y^2 = 1$.

$$x^2 + x^2 = 1 \Longrightarrow \ 2x^2 = 1 \Longrightarrow \ \ x = \pm\frac{\sqrt{2}}{2}.$$

Thus the critical points on the set $\{(x,y) | \ x^2 + y^2 = 1\}$ are:

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

3. Find the value of f at all critical points.

$$f(0,0) = 0$$
$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$
$$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$$
$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$$
$$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}.$$

So the minimum value occurs at $(0,0)$ and the minimum value is 0. The maximum value occurs at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and the maximum value is $\frac{3}{2}$.