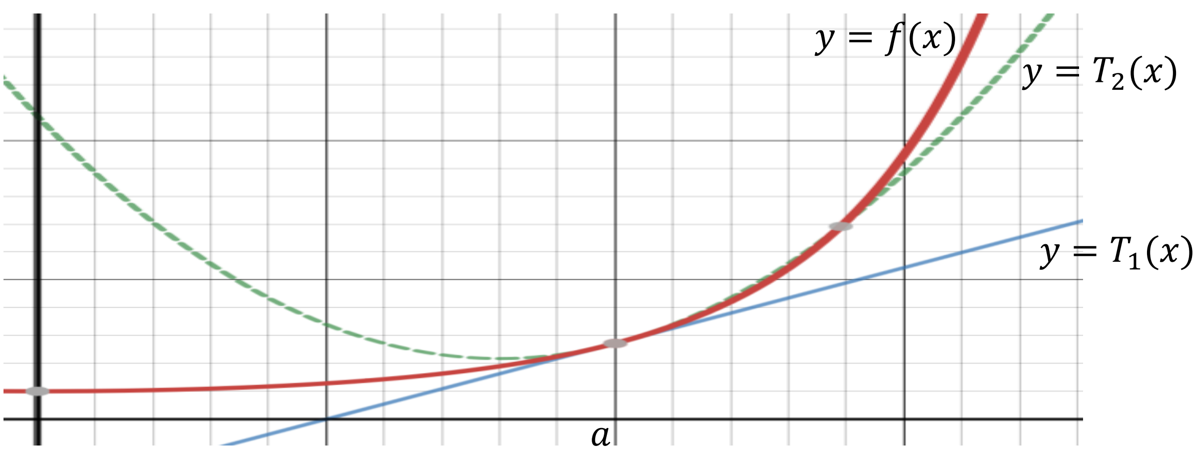
Taylor Series in 2 Variables

Recall from first year calculus that if $f(x)$ has an infinite number of derivatives near a point $x = a$, then we have:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x,a)$$

where $R_n(x,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ and c is between x and a.

This allows us to approximate the value of a function, $f(x)$, using an n-th degree polynomial if we know the value of the function and its derivatives at $x = a$. In addition, $R_n(x,a) =$ error in the approximation, allows us to put an upper bound on the error of this approximation.



where:

$$T_1(x) = f(a) + f'(a)(x-a)$$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2.$$

Ex. Approximate the value of $e^{.02}$ using a second order Taylor polynomial around $a = 0$.

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ f'''(x) &= e^x & f'''(0) &= 1 \end{aligned}$$

$$e^x = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x^2) + R_2(x, 0)$$

where $R_2(x, 0) = \frac{f'''(c)}{3!}(x - 0)^3 = \frac{e^c}{3!}x^3$ and $0 < c < x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{e^c}{3!}x^3$$

So now we can plug in $x = .02$

$$e^{.02} \approx 1 + .02 + \frac{(.02)^2}{2!} = 1 + .02 + \frac{.0004}{2} = 1.0202$$

The error is no bigger than:

$$|R_2(.02, 0)| = \frac{e^c}{3!}(.02)^3 \leq \frac{3}{3!}(.000008) = .000004.$$

Now we want to be able to approximate the value of $f(x, y)$ using polynomials in x and y.

Second Order Taylor Formula:

Let $f: U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ have continuous partial derivatives of third order and let $(x_0, y_0) \in U$. Then we have:

$$f(x,y) = f(x_0,y_0) + \left(f_x(x_0,y_0)\right)(x - x_0) + \left(f_y(x_0,y_0)\right)(y - y_0)$$
$$+\frac{1}{2}\left[f_{xx}(x_0,y_0)(x - x_0)^2 + 2f_{xy}(x_0,y_0)(x - x_0)(y - y_0) +\right.$$
$$\left.f_{yy}(x_0,y_0)(y - y_0)^2\right] + R_2(x,y,x_0,y_0)$$

where:

$$R_2(x,y,x_0,y_0) = \frac{1}{3}\sum_{i,j,k=1}^{2}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(c_{ijk})\right)(\Delta x_i)(\Delta x_j)(\Delta x_k)$$
$$x_1 = x$$
$$x_2 = y$$
$$\Delta x_i = x - x_0 \text{ if } i = 1$$
$$= y - y_0 \text{ if } i = 2$$

and c_{ijk} lies somewhere on the line segment between (x, y) and (x_0, y_0).

Ex. Compute the second order Taylor formula for $f(x,y) = e^x \cos y$ about the point $(x_0, y_0) = (0,0)$ and approximate the value of $e^{.02}\cos(.04)$.

$$
\begin{aligned}
f(x,y) &= e^x \cos y & f(0,0) &= 1\\
f_x &= e^x \cos y & f_x(0,0) &= 1\\
f_y &= -e^x \sin y & f_y(0,0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
f_{xx} &= e^x \cos y & f_{xx}(0,0) &= 1\\
f_{xy} &= -e^x \sin y & f_{xy}(0,0) &= 0\\
f_{yy} &= -e^x \cos y & f_{yy}(0,0) &= -1
\end{aligned}
$$

$$
\begin{aligned}
e^x \cos y &= 1 + 1(x-0) + 0(y-0)\\
&\quad + \tfrac{1}{2}[1(x-0)^2 + 2(0)(x-0)(y-0) - 1(y-0)^2] + R_2(x,y,0,0)\\
&= 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + R_2(x,y,0,0).
\end{aligned}
$$

Thus we have:

$$e^x \cos y \approx 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2$$

$$
\begin{aligned}
e^{.02}\cos(.04) &\approx 1 + .02 + \tfrac{1}{2}(.02)^2 - \tfrac{1}{2}(.04)^2\\
&= 1 + .02 + .0002 - .0008 = 1.0194.
\end{aligned}
$$

Ex. Determine the second order Taylor formula for $f(x,y) = x\cos y$ about the point $(1, \frac{\pi}{4})$ and approximate:

$$f\left(1.1, \frac{\pi}{4} + .2\right) = (1.1)\cos(\frac{\pi}{4} + .2).$$

$$f(x,y) = x\cos y \qquad f\left(1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f_x = \cos y \qquad f_x\left(1, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f_y = -x\sin y \qquad f_y\left(1, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f_{xx} = 0 \qquad f_{xx}\left(1, \frac{\pi}{4}\right) = 0$$

$$f_{xy} = -\sin y \qquad f_{xy}\left(1, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f_{yy} = -x\cos y \qquad f_{yy}\left(1, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\begin{aligned} f(x,y) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-1) - \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) \\ &\quad + \frac{1}{2}\left[0(x-1)^2 + 2\left(-\frac{\sqrt{2}}{2}\right)(x-1)\left(y - \frac{\pi}{4}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\pi}{4}\right)^2\right] \\ &\quad + R_2(x, y, 1, \frac{\pi}{4}) \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-1) - \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) \\ &\quad + \frac{1}{2}\left[-\sqrt{2}(x-1)\left(y - \frac{\pi}{4}\right) + (-\frac{\sqrt{2}}{2})\left(y - \frac{\pi}{4}\right)^2\right] + R_2(x, y, 1, \frac{\pi}{4}). \end{aligned}$$

$$\begin{aligned} &(1.1)\cos\left(\frac{\pi}{4} + .2\right) \\ &\quad \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(.1) - \frac{\sqrt{2}}{2}(.2) + \frac{1}{2}\left[-\sqrt{2}(.1)(.2) + \left(-\frac{\sqrt{2}}{2}\right)(.2)^2\right] \\ &\quad = (.9)\frac{\sqrt{2}}{2} - (.04)\left(\frac{\sqrt{2}}{2}\right) \approx .6081. \end{aligned}$$