Higher Order Partial Derivatives

As with functions of 1 variable, we can take 2nd partial derivatives (and higher order derivatives) of a function $z = f(x, y)$: the partial derivative of a partial derivative.

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Ex. Find the 2nd partial derivatives of $f(x, y) = e^x - 3xy^2 - \sin y$.

$$f_x = e^x - 3y^2 \qquad\qquad f_y = -6xy - \cos y$$

$$f_{xx} = e^x \qquad\qquad f_{yx} = -6y$$

$$f_{xy} = -6y \qquad\qquad f_{yy} = \sin y$$

Notice $f_{xy} = f_{yx}$; this happens often.

Theorem: Suppose f is defined on a disk, D, that contains the point (a, b). If f_{xy} and f_{yx} are both continuous on D, then:

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Ex. Calculate f_{xyxz} if $f(x, y, z) = e^{(x+yz)}$.

$$f_x = e^{x+yz}\frac{\partial}{\partial x}(x + yz) = e^{x+yz}$$

$$f_{xy} = e^{x+yz}\frac{\partial}{\partial y}(x + yz) = e^{x+yz}z$$

$$f_{xyx} = ze^{x+yz}\frac{\partial}{\partial x}(x + yz) = ze^{x+yz}$$

$$f_{xyxz} = z(e^{x+yz})(y) + e^{x+yz} = (zy + 1)e^{x+yz}.$$

Just as we have differential equations that we need to solve that describe processes in the world, for example:

if $f'(t) = kf(t);\ then\ f(t) = ce^{kt}$,

there are partial differential equations that describe processes in the world.

One example is Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solutions of this equation are called harmonic functions, which can be used to describe heat conduction, fluid flow, and electric potential.

Ex. Show $u(x,y) = \ln\sqrt{x^2+y^2}$ satisfies Laplace's equation.

$$u(x,y) = \tfrac{1}{2}\ln(x^2+y^2)$$

$$u_x = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) \qquad u_y = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right)$$

$$u_x = \frac{x}{x^2+y^2} \qquad u_y = \frac{y}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} \qquad u_{yy} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{-x^2+y^2}{(x^2+y^2)^2} \qquad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$u_{xx} + u_{yy} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0.$$

Ex. The wave equation is given by $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ where a is a constant. Suppose f and g are twice differentiable functions of a single variable. Show that: $u = f(v) + g(w)$ where $v = x + at$ and $w = x - at$ is a solution to the wave equation.

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial t} = a\frac{\partial f}{\partial v} - a\frac{\partial g}{\partial w}$$

$$\frac{\partial^2 u}{\partial t^2} = a\left(\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial t}\right) - a\left(\frac{\partial^2 g}{\partial w^2}\frac{\partial w}{\partial t}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 f}{\partial v^2} + a^2\frac{\partial^2 g}{\partial w^2} = a^2\left(\frac{\partial^2 f}{\partial v^2} + \frac{\partial^2 g}{\partial w^2}\right).$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial w}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 g}{\partial w^2}\frac{\partial w}{\partial x} = \frac{\partial^2 f}{\partial v^2} + \frac{\partial^2 g}{\partial w^2}$$

So we have: $$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$$