Directional Derivatives and Gradients

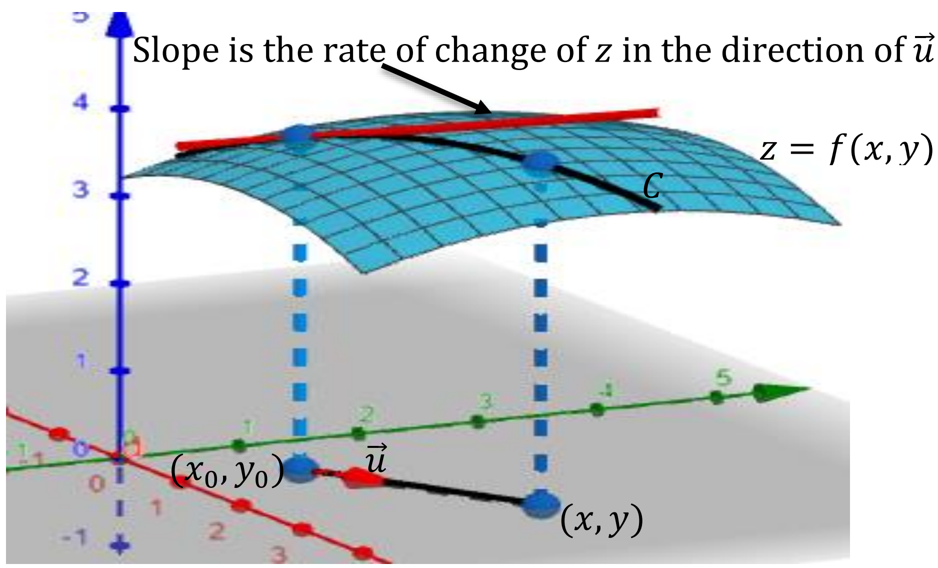
So far, given $z = f(x, y)$, we have the partial derivatives at (x_0, y_0):

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

f_x is the rate of change of z in the x direction (in direction of $\vec{\imath}$) and f_y is the rate of change of z in the y direction (in direction of $\vec{\jmath}$).

How do we find the rate of change of z in the direction of any (unit) vector $< a, b > = \vec{u}$?



Def. The **directional derivative of f** at (x_0, y_0) in the direction of a unit vector, $\vec{u} = < a, b >$, is the following:

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Notice when $\vec{u} = < 1, 0 >$ the definition gives $f_x(x_0, y_0)$
$\vec{u} = < 0, 1 >$ the definition gives $f_y(x_0, y_0)$.

That is: $D_{\vec{i}} f(x_0, y_0) = f_x(x_0, y_0)$
$D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)$.

Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector, $\vec{u} = < a, b >$, and:

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Proof: Define a function of h by $g(h) = f(x_0 + ha, y_0 + hb)$, then by definition

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

$$= D_{\vec{u}} f(x_0, y_0).$$

On the other hand, we can write: $g(h) = f(x, y)$ where

$$x = x_0 + ah$$
$$y = y_0 + bh$$

By the Chain Rule:

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

When $h = 0$; $x = x_0$, $y = y_0$ we get:

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

So we have:

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Notice that:

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = < f_x(x, y), f_y(x, y) > \cdot < a, b >$$

So we can write: $D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$.

Ex. If $f(x,y) = \cos y + xe^y$, find the directional derivative of f in the direction of $\vec{v} = 3\vec{\imath} + 4\vec{\jmath}$ at $(-2,0)$.

$$\nabla f(x,y) = f_x\vec{\imath} + f_y\vec{\jmath} = e^y\vec{\imath} + (-\sin y + xe^y)\vec{\jmath}$$

$$\nabla f(-2,0) = \vec{\imath} - 2\vec{\jmath} = < 1,-2 >$$

If $\vec{v} = 3\vec{\imath} + 4\vec{\jmath}$, then $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$

$$\vec{u} = \frac{3}{5}\vec{\imath} + \frac{4}{5}\vec{\jmath} =< \frac{3}{5}, \frac{4}{5} >$$

$$D_{\vec{u}}f(-2,0) = \nabla f(-2,0) \cdot < \frac{3}{5}, \frac{4}{5} >$$

$$=< 1,-2 > \cdot < \frac{3}{5}, \frac{4}{5} >$$

$$= \frac{3}{5} - \frac{8}{5} = -1.$$

Function of 3 Variables:

$$w = f(x, y, z)$$

Def. The **directional derivative of $\boldsymbol{f}$** at (x_0, y_0, z_0) in the direction of unit vector, $\vec{u} = < a, b, c >$, is defined as:

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah,\ y_0 + bh,\ z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

If the limit exists.

Using vector notation:

$$D_{\vec{u}} f(\overrightarrow{x_0}) = \lim_{h \to 0} \frac{f(\overrightarrow{x_0} + h\vec{u}) - f(\overrightarrow{x_0})}{h}$$

$\overrightarrow{x_0} =< x_0, y_0, z_0 >$ when $n = 3$
$\overrightarrow{x_0} =< x_0, y_0 >$ when $n = 2$.

In 3 variables, if it exists:

$$\nabla f(x, y, z) = f_x(x, y, z)\vec{\imath} + f_y(x, y, z)\vec{\jmath} + f_z(x, y, z)\vec{k}$$

$$= < f_x, f_y, f_z >$$

and:

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}.$$

Ex. If $f(x,y,z) = xe^{yz}$, find the directional derivative at $(2,0,-1)$ in the direction $\vec{v} = -\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$.

$$\nabla f(x,y,z) = f_x\vec{\imath} + f_y\vec{\jmath} + f_z\vec{k} = e^{yz}\vec{\imath} + xze^{yz}\vec{\jmath} + xye^{yz}\vec{k}$$

$$\nabla f(2,0,-1) = \vec{\imath} - 2\vec{\jmath}$$

$$\vec{v} = -\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$$

$$|\vec{v}| = \sqrt{1+9+4} = \sqrt{14}$$

$$\vec{u} = \frac{-1}{\sqrt{14}}\vec{\imath} + \frac{3}{\sqrt{14}}\vec{\jmath} + \frac{2}{\sqrt{14}}\vec{k}$$

$$D_{\vec{u}}f(2,0,-1) = \nabla f(2,0,-1) \cdot \vec{u}$$

$$= (\vec{\imath} - 2\vec{\jmath}) \cdot (\frac{-1}{\sqrt{14}}\vec{\imath} + \frac{3}{\sqrt{14}}\vec{\jmath} + \frac{2}{\sqrt{14}}\vec{k}) = -\frac{1}{\sqrt{14}} - \frac{6}{\sqrt{14}} = -\frac{7}{\sqrt{14}}.$$

Maximizing the Directional Derivative:

Given a function of 2 or 3 variables, we can consider the directional derivative at a point in any direction. We can then ask, in which directions does f change the fastest?

Theorem: Suppose f is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative, $D_{\vec{u}}f(x,y)$ is $|\nabla f(\vec{x_0})|$ and it occurs when $\vec{u}$ has the same direction as $\nabla f(\vec{x})$.

Proof:

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between ∇f and $\vec{u}$.

Hence the maximum occurs when $\cos\theta = 1$ (i.e. $\theta = 0$). Therefore, the maximum value occurs when $\vec{u}$ and ∇f are in the same direction.

Ex. a) Find the rate of change of $f(x,y) = \frac{y^2}{x}$ at the point, $P(2,4)$, in the direction from P to $Q(3,2)$.

b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

a) $\nabla f(x,y) = f_x\vec{\imath} + f_y\vec{\jmath} = -\frac{y^2}{x^2}\vec{\imath} + \frac{2y}{x}\vec{\jmath}$

$\nabla f(2,4) = -\frac{16}{4}\ \vec{\imath} + \frac{8}{2}\vec{\jmath} = -4\vec{\imath} + 4\vec{\jmath}$

$\vec{v} = < 3,2 > - < 2,4 > = \vec{\imath} - 2\vec{\jmath}$

$\vec{v} = \vec{\imath} - 2\vec{\jmath}\,;\ \ so\ \ \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\vec{\imath} - \frac{2}{\sqrt{5}}\vec{\jmath}$

$D_{\vec{u}}f(x,y) = \nabla f(x,y)\ \cdot\ \vec{u} = \left(-\frac{y^2}{x^2}\vec{\imath} + \frac{2y}{x}\vec{\jmath}\right)\ \cdot\ \left(\frac{1}{\sqrt{5}}\vec{\imath} - \frac{2}{\sqrt{5}}\vec{\jmath}\right)$

$D_{\vec{u}}f(2,4) = \left(-4\vec{\imath} + 4\vec{\jmath}\right)\ \cdot\left(\frac{1}{\sqrt{5}}\vec{\imath} - \frac{2}{\sqrt{5}}\vec{\jmath}\right) = -\frac{4}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -\frac{12}{\sqrt{5}}\ .$

b) $\nabla f(2,4) = -4\vec{\imath} + 4\vec{\jmath}$ so f has its maximum rate of change in the direction:

$\vec{v} = -4\vec{\imath} + 4\vec{\jmath}\ \ \Rightarrow\ \ \vec{u} = \frac{-4\vec{\imath}+4\vec{\jmath}}{\sqrt{16+16}} = \frac{-4\vec{\imath}+4\vec{\jmath}}{4\sqrt{2}} = \frac{-1}{\sqrt{2}}\vec{\imath} + \frac{1}{\sqrt{2}}\vec{\jmath}$

$D_{\vec{u}}f(2,4) = \nabla f(2,4)\ \cdot\ \vec{u}$

$D_{\vec{u}}f(2,4) = \left(-4\vec{\imath} + 4\vec{\jmath}\right)\ \cdot\left(\frac{-1}{\sqrt{2}}\vec{\imath} + \frac{1}{\sqrt{2}}\vec{\jmath}\right) = \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{8}{\sqrt{2}}\,.$

So the maximum rate of change is $\frac{8}{\sqrt{2}}$.

Ex. Suppose the temperature at a point in space is given by:

$$T(x,y,z) = 200e^{(-x^2-3y^2-9z^2)}$$

where T is measured in degrees Celsius and x, y, z in meters.

a) Find the rate of change of the temperature at $P(2,-1,2)$ in the direction towards $Q(3,-3,3)$.

b) In which direction does the temperature increase the fastest?

c) What is the maximum rate of increase?

a) $\nabla T = T_x\vec{\imath} + T_y\vec{\jmath} + T_z\vec{k}$

$$= -400xe^{(-x^2-3y^2-9z^2)}\vec{\imath} - 1200ye^{(-x^2-3y^2-9z^2)}\vec{\jmath}$$
$$-3600ze^{(-x^2-3y^2-9z^2)}\vec{k}\,.$$

$$\nabla T(2,-1,2) = -800e^{-43}\vec{\imath} + 1200e^{-43}\vec{\jmath} - 7200e^{-43}\vec{k}$$
$$= 400e^{-43}\left[-2\vec{\imath} + 3\vec{\jmath} - 18\vec{k}\right].$$

$$\vec{v} = \vec{Q} - \vec{P} =< 3-2, -3-(-1), 3-2 > = \vec{\imath} - 2\vec{\jmath} + \vec{k}$$

$$\vec{v} = \vec{\imath} - 2\vec{\jmath} + \vec{k} \qquad \Rightarrow \qquad \vec{u} = \frac{\vec{\imath}-2\vec{\jmath}+\vec{k}}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}\vec{\imath} - \frac{2}{\sqrt{6}}\vec{\jmath} + \frac{1}{\sqrt{6}}\vec{k}$$

$$D_{\vec{u}}T(2,-1,2) = \nabla T(2,1,2)\cdot\vec{u}$$
$$= 400e^{-43}\left[-\frac{2}{\sqrt{6}} - \frac{6}{\sqrt{6}} - \frac{18}{\sqrt{6}}\right] = 400e^{-43}\left[-\frac{26}{\sqrt{6}}\right]$$

b) Max rate of change occurs in direction of

$$\nabla T(2,-1,2) = 400e^{-43}[-2\vec{\imath} + 3\vec{\jmath} - 18\vec{k}].$$

c) Maximum rate of change:

$$\vec{u} = \frac{-2\vec{\imath}+3\vec{\jmath}-18\vec{k}}{\sqrt{4+9+324}} = \frac{-2}{\sqrt{337}}\vec{\imath} + \frac{3}{\sqrt{337}}\vec{\jmath} - \frac{18}{\sqrt{337}}\vec{k}$$

$$D_{\vec{u}}\mathrm{T}(2,-1,2) = 400e^{-43}\left[-2\vec{\imath} + 3\vec{\jmath} + 18\vec{k}\right] \cdot \left[\frac{-2}{\sqrt{337}}\vec{\imath} + \frac{3}{\sqrt{337}}\vec{\jmath} - \frac{18}{\sqrt{337}}\vec{k}\right]$$

$$= 400e^{-43}\left[\frac{4+9+324}{\sqrt{337}}\right] = 400e^{-43}\left[\frac{337}{\sqrt{337}}\right] = 400\sqrt{337}e^{-43}.$$

Tangent Planes for Surfaces $F(x,y,z) = k$

Let $R(t)$ be any differentiable curve on the surface $F(x,y,z) = k$ through (x_0, y_0, z_0).

$$R(t) = < x(t), y(t), z(t) >; \quad R(t_0) = < x_0, y_0, z_0 >$$

Since $R(t)$ is on the surface $F(x,y,z) = k$, we have:

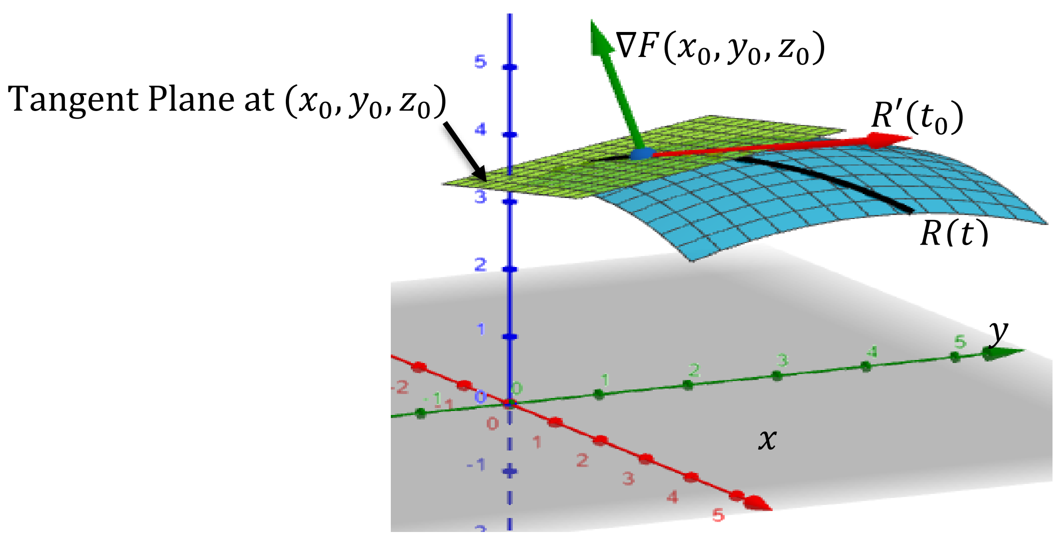
$$F(x(t), y(t), z(t)) = k.$$

Differentiating each side and using the chain rule we get:

$$F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = 0, \text{ so}$$

$\nabla F \cdot R'(t) = 0.$ In particular: $\nabla F(x_0, y_0, z_0) \cdot R'(t_0) = 0.$

So $\nabla F(x_0, y_0, z_0) \perp R'(t_0)$, thus $\nabla F \perp$ surface.



Tangent Plane to $\boldsymbol{F(x,y,z)=k}$ at point $\boldsymbol{(x_0,y_0,z_0)}$:

$$\boldsymbol{F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.}$$

Notice if $z=f(x,y)$, then $F(x,y,z)=f(x,y)-z=0$

$$F_x=f_x\,,\qquad F_y=f_y\,,\qquad F_z=-1$$

Tangent plane: $f_x(x-x_0)+f_y(y-y_0)-(z-z_0)=0$

Or

$$z-z_0=f_x(x-x_0)+f_y(y-y_0).$$

Ex. Find the tangent plane and normal line to the surface

$x^2-2y^2+z^2+yz=2$ at $(2,1,-1)$.

So we have: $F(x,y,z)=x^2-2y^2+z^2+yz=2$

$F_x=2x$ $\qquad F_x(2,1,-1)=2(2)=4$

$F_y=-4y+z$ $\qquad F_y(2,1,-1)=-4(1)-1=-5$

$F_z=2z+y$ $\qquad F_z(2,1,-1)=2(-1)+1=-1$

Equation of tangent plane:

$$4(x-2)-5(y-1)-1(z+1)=0$$

$$4x-8-5y+5-z-1=0$$

$$4x-5y-z=4.$$

$\nabla F(2,1,-1) =< 4,-5,-1 >$ is normal to the surface at $(2,1,-1)$.

Equation of normal line at $(2,1,-1)$:

$$x = 2 + 4t$$

$$y = 1 - 5t$$

$$z = -1 - t.$$

Ex. Find an equation of the tangent plane and the normal line to the surface $x^2yz + 2xz = 12$ at $(1,2,3)$.

So we have: $F(x,y,z) = x^2yz + 2xz = 12$

$F_x = 2xyz + 2z$ $\qquad$ $F_x(1,2{,}3) = 2(6) + 2(3) = 18$

$F_y = x^2z$ $\qquad$ $F_y(1,2{,}3) = 3$

$F_z = x^2y + 2x$ $\qquad$ $F_z(1,2{,}3) = 2 + 2 = 4$

Equation of plane:

$$18(x-1) + 3(y-2) + 4(z-3) = 0$$

Or

$$18x + 3y + 4z = 36.$$

$\nabla F(1,2,3) =< 18,3,4 >$ is normal to the surface at $(1,2,3)$.

Equation of normal line at $(1,2,3)$:

$$\vec{l}(t) =< 1,2,3 > +t < 18,3,4 >$$

$$=< 1 + 18t,\ 2 + 3t,\ 3 + 4t >.$$

Or in parametric form:

$$x = 1 + 18t$$

$$y = 2 + 3t$$

$$z = 3 + 4t.$$