Partial Fractions

Def. A **rational function**, $f(x)$, is a function that can be written as $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Partial fraction is a method to write a rational function as the sum/difference of simpler rational functions.

Ex. Notice the following:

$$\frac{1}{x+3} + \frac{1}{x+2} = \frac{x+2}{(x+3)(x+2)} + \frac{x+3}{(x+2)(x+3)}$$

$$= \frac{2x+5}{x^2+5x+6}.$$

Thus, we can write:

$$\frac{2x+5}{x^2+5x+6} = \frac{1}{x+3} + \frac{1}{x+2}$$

So:

$$\int \frac{2x+5}{x^2+5x+6}\,dx = \int \frac{1}{x+3}\,dx + \int \frac{1}{x+2}\,dx$$

$$= \ln|x+3| + \ln|x+2| + C.$$

Every rational function can be written as a polynomial (possibly the zero polynomial) plus terms of the form:

$$\frac{A}{(ax+b)^i} \quad \text{and} \quad \frac{Ax+B}{(px^2+qx+r)^j}.$$

Partial Fraction Decomposition of $f(x) = \frac{P(x)}{Q(x)}$

1. If the degree of $P(x)$ is bigger than or equal to the degree of $Q(x)$, then divide $Q(x)$ into $P(x)$.

   Ex. $\frac{x^3+x}{x+1}$

$$
\begin{array}{r|l}
 & x^2 - x + 2 - \frac{2}{x+1} \\
\hline
x+1 & x^3 \quad + \quad x \\
 & \underline{x^3 + x^2} \\
 & \quad -x^2 + x \\
 & \quad \underline{-x^2 - x} \\
 & \qquad\quad 2x \\
 & \qquad\quad \underline{2x + 2} \\
 & \qquad\qquad -2
\end{array}
$$

$$\frac{x^3+x}{x+1} = x^2 - x + 2 - \frac{2}{x+1}$$

$$\int \frac{x^3+x}{x+1}\,dx = \int \left(x^2 - x + 2 - \frac{2}{x+1}\right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x - 2\ln|x+1| + C.$$

2. Completely factor the denominator $Q(x)$ into factors of the form $(ax+b)^i$ and $(px^2+qx+r)^j$.

3. For each factor $(ax+b)^i$ the partial fraction decomposition must include the fractions:

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_i}{(ax+b)^i}$$

4. For each factor $(px^2+qx+r)^j$ the partial fraction decomposition must include the fractions:

$$\frac{B_1x+C_1}{px^2+qx+r}+\frac{B_2x+C_2}{(px^2+qx+r)^2}+\cdots+\frac{B_jx+C_j}{(px^2+qx+r)^j}$$

Ex. Evaluate $\int \frac{x-5}{x^2+2x-3}\,dx$.

1. The numerator is of lower degree than the denominator so we don't divide the denominator into the numerator.

2. $\frac{x-5}{x^2+2x-3}=\frac{x-5}{(x+3)(x-1)}$

3. $\frac{x-5}{x^2+2x-3}=\frac{x-5}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}$

We need to solve for A and B. We do this by adding the fractions on the RHS and then setting the numerator equal to $x-5$.

$$\frac{x-5}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}=\frac{A(x-1)}{(x+3)(x-1)}+\frac{B(x+3)}{(x+3)(x-1)}$$

$$x-5=A(x-1)+B(x+3)$$

$$=Ax-A+Bx+3B$$

$$=(A+B)x+(-A+3B).$$

The coefficient of the two polynomials must be equal:

$$1 = A + B$$

$$-5 = -A + 3B.$$

To solve these simultaneous equations for A and B just add the equations.

$$\begin{array}{rcl} 1 = & A + & B \\ -5 = & -A + & 3B \\ \hline -4 = & & 4B \qquad \Rightarrow B = -1. \end{array}$$

Now substitute in the first equation:

$$1 = A + (-1) \qquad \Rightarrow \quad A = 2.$$

$$\frac{x-5}{(x+3)(x-1)} = \frac{2}{x+3} - \frac{1}{x-1}\,.$$

So we have:

$$\int \frac{x-5}{x^2+2x-3}\,dx = \int \frac{2}{x+3}\,dx - \int \frac{1}{x-1}\,dx$$
$$= 2\ln|x+3| - \ln|x-1| + C.$$

Ex. Evaluate $\int \frac{2x^4+4x^2-x-1}{x^3-x}\,dx$.

1. The degree of the numerator is bigger than or equal to the degree of the denominator so divide the denominator into the numerator.

$$\begin{array}{r|l} & \quad 2x \\ \hline x^3 - x & 2x^4 + 4x^2 - x - 1 \\ & 2x^4 - 2x^2 \\ \hline & \qquad\quad 6x^2 - x - 1 \end{array}$$

$$\frac{2x^4+4x^2-x-1}{x^3-x} = 2x + \frac{6x^2-x-1}{x^3-x}\,.$$

2. Factor $x^3 - x$:

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1).$$

3. Add the fractions to solve for A, B, and C

$$\frac{6x^2-x-1}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{A(x-1)(x+1)}{x(x-1)(x+1)} + \frac{B(x)(x+1)}{x(x-1)(x+1)} + \frac{C(x)(x-1)}{x(x-1)(x+1)}$$

$$6x^2 - x - 1 = A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$$6x^2 - x - 1 = (A+B+C)x^2 + (B-C)x + (-A)$$

$$\Longrightarrow \quad \begin{aligned} 6 &= A+B+C \\ -1 &= B-C \\ -1 &= -A \qquad\qquad \Rightarrow A = 1. \end{aligned}$$

Substituting $A = 1$ in the first equation we get:

$$6 = 1 + B + C \quad \Rightarrow \quad \begin{aligned} 5 &= B + C \\ -1 &= B - C \\ \hline 4 &= 2B \qquad \Rightarrow \quad B = 2. \end{aligned}$$

Now, since $5 = B + C \;\Rightarrow\; 5 = 2 + C \;\Rightarrow C = 3$:

$$\frac{6x^2-x-1}{x^3-x} = \frac{1}{x} + \frac{2}{x-1} + \frac{3}{x+1}.$$

$$\int \frac{2x^4+4x^2-x-1}{x^3-x}\,dx = \int \left(2x + \frac{1}{x} + \frac{2}{x-1} + \frac{3}{x+1}\right) dx$$

$$= x^2 + \ln|x| + 2\ln|x-1| + 3\ln|x+1| + C.$$

Now let's do an example with both linear and quadratic factors.

Ex. Evaluate $\int \frac{3x^2-2x+5}{(x+1)(x^2+9)}\,dx$.

The degree of the denominator is larger than the degree of the numerator so don't divide.

$$\frac{3x^2-2x+5}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

Now add the fractions on the RHS:

$$\frac{3x^2-2x+5}{(x+1)(x^2+9)} = \frac{A(x^2+9)}{(x+1)(x^2+9)} + \frac{(Bx+C)(x+1)}{(x+1)(x^2+9)}$$

$$= \frac{(A+B)x^2+(B+C)x+9A+C}{(x+1)(x^2+9)}$$

$$\begin{aligned} 3 &= A + B \quad \Rightarrow \quad B = 3 - A \\ -2 &= B + C \\ 5 &= 9A + C \end{aligned}$$

Substitute $B = 3 - A$ into the second equation, then subtract:

$$-2 = 3 - A + C \quad \Rightarrow \quad \begin{aligned} -5 &= -A + C \\ 5 &= 9A + C \\ \hline -10 &= -10A \quad \Rightarrow \quad A = 1. \end{aligned}$$

$$\begin{aligned} B &= 3 - A = 3 - 1 = 2 \\ -2 &= B + C = 2 + C \\ &\Rightarrow C = -4. \end{aligned}$$

So now we can write:

$$\frac{3x^2-2x+5}{(x+1)(x^2+9)} = \frac{1}{x+1} + \frac{2x-4}{x^2+9}$$

$$\int \frac{3x^2-2x+5}{(x+1)(x^2+9)}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{2x-4}{x^2+9}\,dx$$

$$= \ln|x+1| + \int \frac{2x}{x^2+9}\,dx - \int \frac{4}{x^2+9}\,dx.$$

For the second term:
Let $u = x^2 + 9$

$du = 2x\,dx \quad \Longrightarrow \quad \int \frac{2x}{x^2+9}\,dx = \ln|x^2+9| + C.$

For the third term: $\quad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

so: $\quad 4\int \frac{dx}{x^2+3^2} = \frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$

$$\int \frac{3x^2-2x+5}{(x+1)(x^2+9)}\,dx = \ln|x+1| + \ln|x^2+9| - \frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Ex. Write out the form of the partial fraction decomposition of the following:

$$\frac{2x^3-x^2+x+1}{x^3(x+1)(x^2+x+1)(x^2+3)^2}.$$

$$\frac{2x^3-x^2+x+1}{x^3(x+1)(x^2+x+1)(x^2+3)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{Ex+F}{x^2+x+1} + \frac{Gx+H}{x^2+3} + \frac{Ix+J}{(x^2+3)^2}.$$

Ex. Evaluate $\int \frac{2x^3+x^2+2x+3}{(x^2+1)^2} dx$.

The degree of the denominator is larger than the degree of the numerator do don't divide.

$$\frac{2x^3+x^2+2x+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)}{(x^2+1)^2} + \frac{Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax^3+Bx^2+Ax+B+Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}$$

$$2 = A$$
$$1 = B$$
$$2 = A + C \quad \Rightarrow \quad 2 = 2 + C \quad \Rightarrow \quad C = 0$$
$$3 = B + D \quad \Rightarrow \quad 3 = 1 + D \quad \Rightarrow \quad D = 2.$$

$$\int \frac{2x^3+x^2+2x+3}{(x^2+1)^2} dx = \int \frac{2x+1}{x^2+1} dx + \int \frac{2}{(x^2+1)^2} dx$$

$$= \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx + \int \frac{2}{(x^2+1)^2} dx$$

$$= \ln|x^2 + 1| + \tan^{-1} x + \tan^{-1} x + \left(\frac{x}{x^2+1}\right) + C.$$

To see that $\int \frac{1}{(x^2+1)^2} dx = \frac{1}{2}\tan^{-1} x + \frac{1}{2}\left(\frac{x}{x^2+1}\right)$

Let $x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$$\int \frac{2x^3+x^2+2x+3}{(x^2+1)^2} dx = \ln|x^2+1| + 2\tan^{-1} x + \left(\frac{x}{x^2+1}\right) + C.$$

Ex. Evaluate $\int \frac{4e^x}{e^{2x}-4} dx$.

This integrand can be turned into a rational function by the following substitution:

Let $u = e^x$
$du = e^x\, dx$

$$\int \frac{4e^x}{e^{2x}-4} dx = \int \frac{4}{u^2-4} du$$

$$\frac{4}{u^2-4} = \frac{4}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$= \frac{A(u+2)+B(u-2)}{u^2-4} = \frac{(A+B)u+2A-2B}{u^2-4}$$

$$A + B = 0 \quad \Rightarrow \quad A = -B$$
$$2A - 2B = 4$$
$$2(-B) - 2B = 4$$
$$B = -1 \quad \Rightarrow \quad A = 1.$$

$$\int \frac{4}{u^2-4}\,du = \int \frac{1}{u-2}\,du - \int \frac{1}{u+2}\,du$$

$$= \ln|u-2| - \ln|u+2| + C$$

$$= \ln|e^x-2| - \ln|e^x+2| + C.$$